Algebraic Structures (MST20010) 2023/2024

Problem sheet 2

1. (a) Write the multiplication table of $\mathbb{Z}/3\mathbb{Z}$. Is $(\mathbb{Z}/3Z \setminus \{0\}, \cdot)$ a group (where $\cdot$ denotes the product in $\mathbb{Z}/3\mathbb{Z}$)?

   (b) Is $(\mathbb{Z}/4Z \setminus \{0\}, \cdot)$ a group?

2. Let G be a group, and let $a \in G$. Show that $(a^{-1})^2 = (a^2)^{-1}$.
   Notation: If $b \in G$, by b^k we mean $b \cdot b \cdot \dots \cdot b$ (k times).
   Observe that the same reasoning will give $(a^{-1})^k = (a^k)^{-1}$.

3. Let G be a group.

   (a) Let $a, b, x \in G$. Solve $axb = c$ for x (in terms of a, b and c).

   (b) Let $x, y \in G$ and assume that $xy = yx^2$. Write $xyxy$ in the form $y^i x^j$ for some integers i and j.

4. (This exercise uses Definition 2.12 from the course notes, so you may need to read a little bit more of them.) Let G be a group and let $g \in G$. Let $k = |g|$, the order of g. Show that $g^{mk} = e$ for every $m \in \mathbb{N}$.

   So: If you raise an element to the power of a multiple of its order, you get e. The converse it true (if $g^k = e$ then k is a multiple of the order of g) and will be done in a future exercise sheet.