Problem sheet 11

1. Let G be a finite group with $|G| = p$ with p prime. List all the subgroups of G.

2. Let A_n be the set of all even permutations of S_n. Show that A_n is a subgroup of S_n (we consider that the identity is an even permutation).

3. Let H and K be subgroups of a group G, such that $\gcd(|H|, |K|) = 1$. Show that $H \cap K = \{e\}$. Hint: If $x \in H \cap K$, what can you say about the order of x?

4. Let $K = \{\mathrm{id}, (1\ 2\ 3), (1\ 3\ 2)\} \subseteq S_3$.

   (a) Show that K is a subgroup of S_3. Can you say without computation if K is cyclic?

   (b) How many different left cosets of K are there in S_3? For each of them, give a full list of its elements

   (c) Give a subgroup of S_3 that has exactly 3 different left cosets.

5. We consider the two groups $(\mathbb{R}, +)$ and $(\mathbb{R}^{>0}, \cdot)$ where $\mathbb{R}^{>0} = \{x \in \mathbb{R} \mid x > 0\}$ and $+$, $\cdot$ are the usual operations of sum and product. (You do not have to check that they are groups.)

   Show that the map $\exp : \mathbb{R} \to \mathbb{R}^{>0}$, $\exp(x) = e^x$ (the usual exponential function) is an isomorphism.

It is very convenient to consider a weakening of the definition of isomorphism, where you drop the “bijective” condition:

Let $(G, \cdot)$ and $(H, *)$ be two groups. A map $f : G \to H$ is called a morphism (of groups) if $f(a \cdot b) = f(a) * f(b)$ for every $a, b \in G$. So an isomorphism is a bijective morphism.

6. (a) Show that $f(e_G) = e_H$ and, for every $a \in G$, $f(a^{-1}) = f(a)^{-1}$.

   (b) Show that the set called $\ker f$ (the kernel of f), defined by

   $$\ker f = \{x \in G \mid f(x) = e_H\}$$

   is a subgroup of G.