Problem sheet 9 - Solution

1. (a) Write J for $J(R)$. We prove both inclusions.

   Let $x+I \in J(R/I)$. Then, if $z+I \in R/I$, then $(1+I)-(z+I)(x+I)$ is left-invertible in R/I, i.e. there is $b+I$ such that $(b+I)[(1+I)-(z+I)(x+I)] = 1+I$, i.e. $b(1-zx)+I = 1+I$, i.e. $b(1-zx) = 1+i$ for some $i \in I$. But $1+i$ is left-invertible in R (since $I \subseteq J(R)$), so there is $a \in R$ such that $a(1+i) = 1$, so $ab(1-zx) = 1$, which proves that $1-zx$ is left-invertible. Since it happens for every $z \in R$, we have $x \in J$, so $x+I = J/I$.

   Let $x \in J$, so that $x+I \in J/I$. We show that $x+I \in J(R/I)$ by showing that $(1+I)-(z+I)(x+I)$ is left invertible, for every $z \in R$. Let $z \in R$. Then $1-zx$ is left invertible: There is $a \in R$ such that $a(1-zx) = 1$. Therefore $a(1-zx)+I = 1+I$, i.e., $(a+I)((1+I)-(z+I)(x+I)) = 1+I$, done.

   (b) Simply use the description of the elements of the Jacobson radical in terms of left-invertibility and the fact that the sums and products of elements in $\prod_{i\in I} R_i$ is done coordinate by coordinate.

   (c) We use again the characterization in terms of invertibility. Let $x \in J(R)$. We show that $1+uf(x)$ is left invertible for every $u \in S$. Since f is surjective $u = f(\alpha)$, so $1+uf(x) = f(1+\alpha x)$. Since $x \in J(R)$, $1+\alpha x \in U(R)$, so $f(1+\alpha x) \in U(S)$ (easy to check, do it).

2. I write again J for $J(R)$. There is $z \in R$ such that $(a+J)(z+J) = (z+J)(a+J) = 1+J \in R/J$. So $az = 1+i$ for some $i \in J$. By the properties of J, $1+i$ is invertible in R, so $az(1+i)^{-1} = 1$: a has a right inverse. Similarly a has a left inverse.

3. (a) I is clearly closed by sums, and contains 0. Let $a = (a_n)_{n\in\mathbb{N}} \in I$. Say that $n_1 < \ldots < n_k$ are the coordinates for which $a_i \neq 0$. We have $a_{n_i} = 2.b_{n_i}$ for $i = 1,\ldots,k$ (because $a_{n_i} \in 2.R_{n_i}$. Let $r = (r_n)_{n\in\mathbb{N}} \in R$. Then $ra = (ra_n)_{n\in\mathbb{N}} \in I$, because the only nonzero coordinates are $n_1,\ldots,n_k$ and $r_{n_i}a_{n_i} = 2r_{n_i}b_{n_i}$ for $i = 1,\ldots,k$. We have $ar = ra \in I$.
   So I is an ideal.

   (b) Let $a = (a_n)_{n\in\mathbb{N}} \in I$ with notations as above. If $i \in \{1,\ldots,k\}$ we have $a_{n_i} = 2.b_{n_i} \in R_{n_i}$ and then $a_{n_i}^{n_i} = 0$. Since n_k is the greatest of the n_i we have $a_{n_i}^{n_k} = 0$ for every $i = 1,\ldots,k$. This gives $a^{n_k} = 0$.
   We remark that the exponent n_k depends on a.

   (c) Suppose $I^n = \{0\}$ for some integer n. This implies $a^n = 0$ for every $a \in I$. But if $a = (0,\ldots,0,2,0,\ldots)$, where the 2 is at the coordinate $n+1$ (*i.e.* belongs to $R_{n+1} = \mathbb{Z}/2^{n+1}\mathbb{Z}$), we have $a^n = (0,\ldots,0,2^n,0,\ldots) \neq 0$.

We can deduce that R is not Artinian, because in an Artinian ring a nil ideals is nilpotent.
Define $I_n = \prod_{i=1}^n R_i \times \prod_{i \geq n+1} \{0\}$. The I_n form a strictly ascending chain of ideals of R.
Define $J_n = \prod_{i=1}^n \{0\} \times \prod_{i \geq n+1} R_i$. The J_n form a strictly descending chain of ideals of R.

4. (a) Clear.

   (b) Same as above, just do it coordinate by coordinate.

   (c) If f is such a function, take for g the function equal to $f(x)^{-1}$ if $f(x) \neq 0$ and 0 is $f(x) = 0$. Then $f = fgf$.

   (d) $(ba)^2 = baba = ba$. Let Ra be a principal left ideal. We show that $Ra = R(ba)$. Obviously $R(ba) \subseteq Ra$. Let now $ra \in Ra$. Then $ra = ra(ba) \in R(ba)$.