Problem sheet 8 - Solution

1. (a) Use the first isomorphism theorem, together with the morphism of R-modules

$$f : N \oplus P \to P, \quad f(n+p) = p,$$

where $n \in N$ and $p \in P$.

(b) "$\Rightarrow$" Let N be a submodule of M. Take $a_1 \in N$. Then $\mathrm{Span}(a_1) \neq N$, take $a_2 \in N \backslash \mathrm{Span}(a_1)$. Then $\mathrm{Span}(a_1, a_2) \neq N$, etc. We get in this way an infinite strictly increasing chain of submodules of N (hence of M), contradition.

"$\Leftarrow$" Let $N_0 \subseteq N_1 \subseteq \cdots$ be an increasing chain of submodules of M. Then $\bigcup_{i\geq 0} N_i = \mathrm{Span}(m_1, \ldots, m_k)$ for some $m_1, \ldots, m_k \in M$ and since there is $j \in \mathbb{N}$ such that $m_1, \ldots, m_k \in N_j$ we get $\bigcup_{i\geq 0} N_i = N_j$, and so $N_j = N_{j+1} = \cdots$.

2. (a) Let $\{e_i\}_{i\in I}$ be a basis of M, and fix $i_0 \in I$, $m \in M \setminus \{0\}$. Define $f \in \mathrm{End}_D M$ by $f(e_{i_0}) = m$ and $f(e_i) = 0$ if $i \neq i_0$. We have $\dim_D \mathrm{Im}(f) = 1$, so $f \in I$, which is then nonzero.
Let $f \in I$ and $g \in \mathrm{End}_D M$. Then $\mathrm{Im}(f \circ g) = f(g(M)) \subseteq f(M)$ which has finite dimension, and $\mathrm{Im}(g \circ f) = g(f(M))$, which has finite dimension since $\dim_D \mathrm{Im}\, f$ is finite. This proves $f \circ g, g \circ f \in I$.
If $f, g \in I$, then $Im(f+g) = (f+g)(M) \subseteq f(M) + g(M)$ which has finite dimension since $\dim f(M)$ and $\dim g(M)$ are finite.
I is then a nonzero ideal of $\mathrm{End}_D M$. I is proper because $\mathrm{Id} \notin I$.

(b) Let $\{e_i\}_{i\in\mathbb{N}}$ be linearly independent subset of M, and let N_k be the submodule generated by $e_1, \cdots, e_k$. Define $L_k = \{f \in \mathrm{End}_D M \mid f(N_k) = \{0\}\}$. L_k is a left ideal of $\mathrm{End}_D M$, and since the N_k form a strictly ascending chain, the L_k form a strictly descending chain. This proves that $\mathrm{End}_D M$ does not satisfy the DCC on left ideals.
We can also show that $\mathrm{End}_D M$ does not satisfy the ACC on left ideals (i.e., is not Noetherian):
Let M_k be the submodule generated by $\{u_i\}_{i\geq k}$, and define

$$J_k = \{f \in \mathrm{End}_D M \mid f(M_k) = \{0\}\}.$$

J_k is a left ideal, and the J_k form a strictly ascending chain.

3. (a) Let $x \in N_2 \setminus N_1$. Then $f(x) \in f(N_2) \setminus f(N_1)$. Indeed: if $f(x) \in f(N_1)$, then $f(x) = f(y)$ for some $y \in N_1$, so $x = y \in N_1$, contradiction.

(b) Assume that f is not surjective, i.e., $f(M) \subsetneqq M$. By (a) we get an infinite strictly descending chain of submodules of M:

$$M \supsetneqq f(M) \supsetneqq f^2(M) \supsetneqq \cdots$$

contradiction.

4. The sets I_n are clearly left ideals, and form an infinite strictly decreasing chain, so R is not Artinian, contradiction.