Problem sheet 4 - Solution

1. (a) Direct.
   (b) Let $a \in \mathrm{Ann}_R(N)$ and $r \in R$. We show that $ar \in \mathrm{Ann}_R N$: Let $n \in N$. Then $arn = a(rn) = 0$ since $rn \in N$.

2. Let $a_1, \dots, a_n$ be generators of N and $b_1 + N, \dots, b_k + N$ be generators of M/N (with $b_1, \dots, b_k \in M$.
   Let $x \in M$. Then $x + N \in M/N$, so there are $r_1, \dots, r_k \in R$ such that $x + N = r_1(b_1+N)+\cdots+r_k(b_k+N) = (r_1b_1+\cdots+r_kb_k)+N$. By definition of the quotient, there is $y \in N$ such that $x = r_1b_1+\cdots+r_kb_k+y$. Since $y \in N$ there are $s_1, \dots, s_n \in R$ such that $y = s_1a_1 + \cdots + s_na_n$, so $x = r_1b_1 + \cdots + r_kb_k + s_1a_1 + \cdots + s_na_n$. So M is generated by the elements $a_1, \dots, a_n, b_1, \dots, b_k$.

3. In order to use the first isomorphism theorem, we must have a morphism of R-modules f and then we obtain an isomorphism from: the quotient of the module where f is defined by the kernel of f, to: the image of f.

   Since the quotient we want to obtain is M/N, we need f to be defined on M, and to have kernel N. We try:

   $$f : M = N \oplus P \to P,\ f(n + p) = p$$

   (where $n \in N$ and $p \in P$).

   It is easy to check that f is a morphism of R-modules (write down the details, ask me if you have a problem), and the image of f is P. We determine $\ker f = \{x \in M \mid f(x) = 0\}$. Let $x = n + p \in M$ (again, $n \in N$, $p \in P$). Then $f(x) = 0$ iff $p = 0$ iff $x = n$ iff $x \in N$. So $\ker f = N$, and the result follows (by the first isomorphism theorem).

4. (a) Let $a = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$. Then a^n has diagonal (x^n, z^n). To get $x^n = 0$ and $z^n = 0$ we must have $x \in 2(\mathbb{Z}/4\mathbb{Z})$ and $z = 0$. Then any value of y will be fine (we get $a^2 = 0$ no matter what y is). So the elements of N have the given form. Conversely, it is easy to see that the elements given are all nilpotent.
   (b) Let $r = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{Ann}_R N$. Then, for every $n = \begin{pmatrix} 2x & y \\ 0 & 0 \end{pmatrix} \in N$ (with $x \in \mathbb{Z}/4\mathbb{Z}$) we have

   $$rn = \begin{pmatrix} 2ax & ay \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we want $2ax = 0$ and $ay = 0$ for every x and y. Therefore $a \in 2(\mathbb{Z}/4\mathbb{Z}))$, and we have no condition on b or c. So

$$\operatorname{Ann}_R N = \begin{pmatrix} 2(\mathbb{Z}/4\mathbb{Z}) & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}.$$

Note that is had 8 elements.

(c) Let $r = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ be in the right annihilator of N in R. Then, for every $n = \begin{pmatrix} 2x & y \\ 0 & 0 \end{pmatrix} \in N$ (with $x \in \mathbb{Z}/4\mathbb{Z}$) we have

$$nr = \begin{pmatrix} 2xa & 2xb + yc \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we want $2ax = 0$ and $2xb + yc = 0$ for every x and y. Therefore the first equation gives $a \in 2(\mathbb{Z}/4\mathbb{Z}))$, and the second first gives $c = 0$ (take $x = 0$, $y = 1$) and thus $2xb = 0$, which is always true (recall that the second equation is in $\mathbb{Z}/2\mathbb{Z}$). There is no condition on b. So

$$\operatorname{Ann}_R^{\text{right}} N = \begin{pmatrix} 2(\mathbb{Z}/4\mathbb{Z}) & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{pmatrix}.$$

Note that it has 4 elements.

(d) Assume that $f : R \to R^{op}$ is an isomorphism. Then $f(N) = N$ (easy, do it) and it follows that $f(\operatorname{Ann}_R N) = \operatorname{Ann}_R^{\text{right}} N$, which is impossible since they do not have the same number of elements.