Problem sheet 3 - Solution

1. (a) If $u = 0$ it is clear. If $u \neq 0$ you can proceed in two ways:
   1) Explicitely: Write $A = (a_{ij})$, $u = (u_1, \ldots, u_n)$. You want to solve $a_{11}u_1 + \cdots + a_{1n}u_n = 0$. That will give you the first row of A, and obviously you can take all rows of A to be equal. This equation has a solution since not all u_i are 0 (there are two cases, if only one u_i is non-zero, and if more than one is non-zero).
   2) Simpler but more abstract: There is a non-zero linear map that sends u to 0 (take a basis containing u, then define the linear map by its values on the elements of the basis). The correspondance between matrices and linear maps gives a matrix A such that $Au = 0$.
   This reasoning can be used to show that if $u \in \mathbb{R}^n \setminus \{0\}$ and $v \in \mathbb{R}^n$, there is $A \in M_n(\mathbb{R})$ such that $Au = v$. It is useful for question (c).
   (b) By (a), any set of elements that contains a non-zero element is linearly dependent.
   (c) To show that $\mathbb{R}^n$ is a simple $M_n(\mathbb{R})$-module, we show that if N is a non-zero submodule of $\mathbb{R}^n$, then $N = \mathbb{R}^n$. Let $u \in N \setminus \{0\}$. If v is any element of $\mathbb{R}^n$ there is a linear map f such that $f(u) = v$. In terms of matrices is means that there is a matrix B such that $Bu = v$. Therefore $\text{Span}(u) = \mathbb{R}^n$. Since $\text{Span}(u) \subseteq N$ we get $N = \mathbb{R}^n$.

2. (a) Clear.
   (b) Because each P_r only involves a finite number of inderterminates.
   (c) Follows from (b) and the definition of N.
   (d) Take f such that $f(X_1) = \cdots = f(X_N) = 0$ and $f(X_{N+1}) = 1$. Since $X_{N+1} \in \text{Span}\{X_1, \ldots, X_N\}$ we have $f(X_{N+1}) = 0$, a contradiction.

3. (a) It is clearly non-empty and closed under sums. Let $r \in R$ and $em \in eM$ (with $m \in M$). Then $r(em) = (re)m = (er)m = e(rm) \in eM$.
   (b) We first show that $M = eM + (1-e)M$ and then that $eM \cap (1-e)M = \{0\}$:
   Let $m \in M$. Then $m = 1 \cdot m = (e + (1-e))m = em + (1-e)m$, so every element can be written as the sum of an element in eM and an element in $(1-e)M$.
   Let $m \in eM \cap (1-e)M$, i.e. $m = em_1 = (1-e)m_2$ (with $m_1, m_2 \in M$). Multiplying by e we get: $e^2m_1 = (e - e^2)m_2$, so $em_1 = 0$, and thus $m = 0$.