# Solutions (problem sheet 1)

Warning: a solution provided here is just this: a solution. Yours is probably quite good or even better.

1. (a) By hypothesis there is $b \in R$ such that $ba^n = 1$ and $a^n b = 1$. It follows that $(ba^{n-1})a = 1$, so $a$ has a left inverse, and $a(a^{n-1}b) = 1$ so $a$ has a right inverse. So $a$ is invertible.

   (b) Let $b$ be a left inverse of $a$. Assume $ac = 0$ for some $c \neq 0$ (i.e. $a$ is a left zero divisor). Multiplying on the left by $b$ we obtain $c = 0$, a contradiction.

   (c) Since $a \in aRa$ we have $a = ara$ for some $r \in R$. Then $a(1-ra) = 0$, and since $a$ is not a left zero divisor, it follows that $1 - ra = 0$, so $ra = 1$, and $r$ is a left inverse of $a$.

2. Consider $(x+x)^2$. By hypothesis we have $(x+x)^2 = x+x$. Developping we get $4x^2 = 2x$ and using the hypothesis again $4x = 2x$, so $2x = 0$, in other words $x = -x$ for every $x \in R$.

   We have $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$. This gives $0 = xy + yx$, *i.e.* $xy = -yx$. Now use that we just proved that $-yx = yx$.

3. We check that $I \cap J$ is a left ideal by using the reformulation of the definition in Remark 1.10 (it is almost always the most convenient).

   To show that $I \cap J$ is non-empty, we observe that it contains 0: We have $0 \in I \cap J$ since $0 \in I$ and $0 \in J$.
   Let $x, y \in I \cap J$, i.e. $x, y \in I$ and $x, y \in J$. Since $I$ and $J$ are left ideals then $x + y \in I$ and $x + y \in J$, so $x + y \in I \cap J$.
   Let $x \in I \cap J$ and $a \in R$. Since $I$ and $J$ are left ideals we have $ax \in I$ and $ax \in J$, so $ax \in I \cap J$.

4. Since $f(0_R) = 0_S$ and $0_S \in K$ we have $0_R \in f^{-1}(K)$.

   Let $a, b \in f^{-1}(K)$ (i.e., $f(a), f(b) \in K$) and let $r \in R$. Then $f(a+b) = f(a) + f(b) \in K$ (since $K$ is closed under sums), which means $a + b \in f^{-1}(K)$, and $f(ra) = f(r)f(a) \in K$ (since $K$ is closed under products on the left), which means $ra \in f^{-1}(K)$. Similarly $ar \in f^{-1}(K)$.

5. (a) The element 1 is the only element with the property that $1 \cdot a = a \cdot 1 = a$ for every $a \in R$. Here, using the definition of the product, the element 1 is characterised by the property $1 \circ f = f \circ 1 = f$

for every $f \in R$. So $1 = \mathrm{id}$ (id satisifies this property, and it is easy to check that if two elements have it, they must be equal).

(b) We want $g \circ (f + h) = g \circ f + g \circ h$ and $(g + f) \circ h = g \circ h + f \circ h$. These are equalities between functions, so to check them, we check them at each point. Let $x \in V$. Then

$$g\circ(f+h)(x) = g(f(x)+h(x)) = g(f(x))+g(h(x)) = g\circ f(x)+g\circ h(x),$$

$$(g+f)\circ h(x) = (g+f)(h(x)) = g(h(x))+f(h(x)) = g\circ h(x)+f\circ h(x).$$

6. We obviously have $g \circ f = \mathrm{id}$, so $g$ is right invertible and $f$ is left invertible. But $g$ is not left invertible, otherwise $g$ would be injective, which is not the case. For a similar reason, $f$ is also not right invertible.