Ring Theory (MATH40630) 2022/2023

Midterm exam

- Put your student number on top of each sheet.
- Start your paper by writing the following statement:
  The following is my own work, and I have not received any help during this exam.
- This midterm will be graded out of 120, but grades will be capped at 100. Each exercise brings 30%.

Observe (no need to prove it) that in the formula

$$\sum_{\substack{i+j=n,\\ i,j\geq 0}} \binom{i}{n} a^i b^j$$

in each term $a^i b^j$ we have $i \geq \frac{n}{2}$ or $j \geq \frac{n}{2}$.

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1. Let R be a commutative ring. Define
$$I = \{r \in R \mid r^n = 0 \text{ for some } n \in \mathbb{N}\}.$$
Show that I is an ideal of R.

2. Let R be a ring in which every non-zero element r has a right inverse denoted r'. Let $x \in R \setminus \{0\}$.
   (a) Show that $(x'x)^2 = x'x$.
   (b) Deduce that $x'x = 1$.
   (c) Show that R is a division ring.

3. Let R be a ring.
   (a) Let $r \in R$. Show that r has a right inverse if and only if $rR = R$.
   (b) Show that R is a division ring if and only if the only two right ideals of R are $\{0\}$ and R. Hint: The result of exercise 2 may be useful.

4. Let R be a ring and let M be an R-module such that $M = \text{Span}_R\{m_0\}$ for some $m_0 \in M$. Show that there is a left ideal L of R such that $R/L \cong M$ (isomorphism as R-modules).