Ring Theory (MATH40630) 2022/2023

Midterm exam

1. I is non-empty since it contains 0.

   Let $r, s \in I$, so $r^n = 0$ and $s^m = 0$ for some $n, m \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $k/2 \geq n, m$. Then

   $$(r+s)^k = \sum_{i=0}^{k} \binom{n}{i} r^i s^{k-j}.$$

   As observed at the start of the midterm, in each term we have $i \geq n$ or $k - j \geq m$. So $(r+s)^k = 0$ and $r + s \in I$. Finally, if $x \in R$, we have $(xr)^n = x^n r^n = 0$ (since R is commutative).

2. (a) $(x'x)^2 = x'(xx')x = x'x$.

   (b) Multiply both sides of the precious equality on the right by the right inverse of $x'x$. However, this is only possible if $x'x$ is not 0, so we must check this: Assume that $x'x = 0$. Therefore $xx'x = 0$ and thus $x = 0$, contradiction.

   (c) The previous question shows that x' is a left and right inverse of x, so every non-zero element has an inverse.

3. (a) "$\Rightarrow$" Let s be a right inverse of r. Then $1 = rs \in rR$, which implies (as seen in class) that $rR = R$. You can give the details of this final bit if you prefer: If $x \in R$, then $x = rsx \in rR$.

   "$\Leftarrow$" Then $1 \in rR$, so there is $s \in R$ such that $rs = 1$.

   (b) "$\Rightarrow$" Let I be a non-zero ideal of R. Take $r \in I \setminus \{0\}$. Then $1 = x^{-1}x \in I$, so $I = R$.

   "$\Leftarrow$" By exercise 2, it suffices to show that every non-zero element has a right inverse. By the first question it suffices to show that $rR = R$ for every $r \in R \setminus \{0\}$. Since $r \neq 0$, we have $rR \neq \{0\}$ and thus $rR = R$ by hypothesis.

4. Define $f : R \to M$, $f(r) = rm_0$. f is a morphism of R-modules (indeed $f(x+y) = f(x) + f(y)$ and $f(rx) = rf(x)$). Since $M = \mathrm{Span}_R\{m_0\}$ we have $\Im f = M$. Finally $\ker f$ is a submodule of R, which is the same as a left ideal (if you prefer you can just check that $\ker f$ is a left ideal). By the first isomorphism theorem (for modules) we have $R/\ker f \cong M$.