Problem sheet 7

1. Full version of Q2 from the 2021 midterm:

   Let R be a ring without zero divisors. Suppose that R has a minimal non-zero left ideal I (we may have $I = R$).

   (a) Show that, for every $x \in R$, if $x \neq 0$ then $x^2 \neq 0$.

   (b) Let $i_0 \in I \setminus \{0\}$. Show that $Ri_0^2 = I$.
   (Where $Ri_0^2 = \{ri_0^2 \mid r \in R\}$.)

   (c) Show that $I = R$. Hint: i_0 belongs to I.

   (d) Show that any non-zero element a in $I = R$ is left invertible (you could consider the left ideal $R.a$) and right invertible.
   So R is a division ring.

2. Full version of Q3(b) from the 2021 midterm:

   Let M be an R-module, and let N be a submodule of M. Show that M is Artinian if and only if both N and M/N are Artinian.

3. Show that a semisimple module is of finite length (i.e. has a composition series) if and only if it is finitely generated.

4. Let M be a semisimple R-module and let U, V be submodules of M such that $M = U + V$.

   (a) Show that there exist two submodules N, P of M such that $U = (U \cap V) \oplus N$ and $V = (U \cap V) \oplus P$.

   (b) Show that $M = N \oplus (U \cap V) \oplus P$.

   We now assume that M has finite length. If W is a module of finite length, we denote by $\ell(W)$ the length of W.

   (c) Show that if W_1, W_2 are submodules of M such that $W_1 + W_2 = W_1 \oplus W_2$, then $\ell(W_1 \oplus W_2) = \ell(W_1) + \ell(W_2)$.

   (d) Deduce that $\ell(U + V) = \ell(U) + \ell(V) - \ell(U \cap V)$.

   Recall that the dimension works in this way in vector spaces (when it is finite): If U and V are finite-dimensional subspaces of vector space W, then

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$