Problem sheet 4
For Wednesday 18 February

1. Let M be an R-module. If N is a subset of M we define the annihilator of N in R:
$$\mathrm{Ann}_R(N) = \{r \in R \mid rx = 0 \text{ for every } x \in N\}.$$
   (a) Show that $\mathrm{Ann}_R(N)$ is a left ideal of R.
   (b) Show that, if N is a submodule of M, then $\mathrm{Ann}_R(N)$ is a two-sided ideal of R.

2. Let M be an R-module and let N be a submodule of M. Show that if N and M/N are both finitely generated as R-modules, then M is finitely generated.

3. Let M be an R-module and let N, P be submodules of M such that $M = N \oplus P$. Show that $M/N \cong P$ as R-modules. Hint: First isomorphism theorem.

4. (This exercise uses the opposite ring of R, introduced in Definition 2.19.)
   We consider the ring
$$R = \begin{pmatrix} \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{pmatrix}.$$
   (The sum is as usual for matrices: coordinate by coordinate. For the product: You do the product as usual in $\mathbb{Z}$ then take the image of the result in the the corresponding quotient of $\mathbb{Z}$. For instance:
$$\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$
   It does indeed give a ring; the product is well-defined because 4 is a multiple of 2, check it if you want.)
   We say that an element a in a ring is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$.
   (a) Show that the set N of nilpotents elements in R is
$$\begin{pmatrix} 2(\mathbb{Z}/4\mathbb{Z}) & \mathbb{Z}/2\mathbb{Z} \\ 0 & 0 \end{pmatrix}.$$
   (b) Compute the annihilator of N in R.
   (c) Compute the right annihilator of N in R (the right annihilator of a subset B of R is $\{r \in R \mid B \cdot r = \{0\}\}$).
   (d) Deduce that R and R^{op} are not isomorphic.