Problem sheet 2

1. Let F be a field and let $A \in M_n(F)$ be left invertible with left inverse B (i.e. $BA = I_n$). Define the maps $\lambda : M_n(F) \to M_n(F)$, $\lambda(X) = AX$.

   (a) Show that λ is an bijective linear map from the F-vector space $M_n(F)$ to itself. Hint: Show that it is injective.

   (b) Deduce that $AB = I_n$ (hint: observe that I_n is in the image of λ).
   So left invertible implies invertible in $M_n(F)$ (the same would work for right invertible).

   Remark: We saw in the previous exercise sheet an example of a ring where some elements are left invertible but not right invertible. This ring was the endomorphim ring of some infinite-dimensional vector space. The reasonning above shows that this cannot happen in the endomorphism ring of a finite-dimensional vector space (because it is isomorphic to $M_n(F)$).

2. Let R be a ring and let X be a nonempty subset of R. Recall that (X) is the intersection of all ideals containing X (which makes it the smallest ideal containing X). Show that

$$(X) = \{\sum_{i=1}^{n} r_i x_i s_i \mid n \in \mathbb{N},\ x_i \in X,\ r_i, s_i \in R\}.$$

   It helps justify the terminology that (X) is generated by X: It is exactly like the set of all linear combinations of elements of X (that you used in Linear Algebra), except that we do products on the left and on the right.

3. Let I be an non-zero ideal of $\mathbb{Z}$ (observe that there is no difference here between left, right and 2-sided ideals, since the product is commutative), and let a be the smallest positive element of I.

   (a) For every element $x \in I$ consider the Euclidean division of x by a (i.e. $x = qa + r$ with $q \in \mathbb{Z}$ and $r \in \{0, \dots, a-1\}$), and deduce that a divides x.

   (b) Conclude that $I = a \cdot \mathbb{Z}$ (the set of all multiples of a).

   Thus, an ideal of $\mathbb{Z}$ is always of the form $a\mathbb{Z}$ for some $a \in \mathbb{Z}$.

4. Let $R = M_n(\mathbb{R})$ and let I be the set of matrices in R with 0 outside of the first column. Show that I is a left ideal of R. Explain how it gives a counter-example to a statement similar to Proposition 1.20 but for left ideals.

5. If R is a ring, the centre of R is

$$C(R) = \{x \in R \mid xy = yx \text{ for all } y \in R\}.$$

Remarks: (1) The centre $C(R)$ of R is often denoted $Z(R)$ (for historical reasons, because of "Zentrum" in German, meaning "centre").
(2) $C(R)$ is always a subring of R. The proof is just a straightforward verification, so we skip it.

Determine $C(\mathbb{H})$, the centre of the ring of the real quaternions.

6. Let R be a ring.

   (a) Assume that the only left or right ideals of R are $\{0\}$ and R. Show that R is a division ring. Hint: You can consider the left ideal generated by an element.

   (b) Show that if R is simple (we say that R is simple if the only 2-sided ideals of R are $\{0\}$ and R) then the centre (cf. question 5) of R is a field. You can use without checking it that $C(R)$ is a subring of R.