Problem sheet 1

1. Let R be a ring and let $a \in R$.
   (a) Let $n \in \mathbb{N}$. Show that if a^n has an inverse, then a has an inverse.
   (b) Show that if a has a left inverse, then a is not a left zero divisor (i.e., if $ab = 0$ then $b = 0$).
   (c) Let $a \in R \setminus \{0\}$. Show that if $a \in aRa$ and a is not a left zero divisor, then a has a left inverse. Notation: $aRa := \{ara \mid r \in R\}$.

2. Let R be a ring in which $x^2 = x$ for all x in R (such a ring is called a Boolean ring). Show that $x + x = 0$ for every $x \in R$, then prove that R is commutative. Hint: You can consider elements of the form $x + y$.

3. Let R be a ring and let I and J be left ideals of R. Show that $I \cap J$ is a left ideal of R. Remark: The same result holds for intersections of any number of left ideals (even infinite), for right ideals and for 2-sided ideals.

4. Let $f : R \to S$ be a morphism of rings (so R and S are rings). Let K be an ideal of S. Show that $f^{-1}(K)$ is an ideal of R.

   (Recall that the definition of $f^{-1}(K)$ is $\{x \in R \mid f(x) \in K\}$. Despite the notation, it does not require f to be invertible, and it coincides with f^{-1} applied to K when f is invertible.)

5. Let V be an F-vector space (where F is a field) and let $\mathrm{End}_F(V)$ be the set of all F-linear maps from V to V. Consider on $\mathrm{End}_F(V)$ the following sum and product (for $f, g \in \mathrm{End}_F(V)$):

   - Sum: the usual sum of maps (so that $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$ for every $x \in V$).
   - Product: the composition of maps. In other words, $fg(x)$ is defined by $(fg)(x) \stackrel{\text{def}}{=} (f \circ g)(x) = f(g(x))$ for every $x \in V$.

   Equipped with these, $\mathrm{End}_F(V)$ is a ring (and a very important one that we will see again) called the endomorphism ring of V. We are not going to check all the properties, just two of them:

   (a) Check that the identity map is the element 1 in the ring $\mathrm{End}_F(V)$.
   (b) Check the property of distributivity.

   Hint: The elements of $\mathrm{End}_F(V)$ are functions. So, two elements $f_1, f_2 \in \mathrm{End}_F(V)$ are equal if and only if $f_1(x) = f_2(x)$ for every $x \in V$.

One more exercise on the other side

6. Example of a ring where some elements are invertible on one side but not on the other. Recall that
$$\mathbb{R}^{\mathbb{N}} = \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{R}\},$$
the set of sequences of real numbers indexed by elements of $\mathbb{N}$, is an $\mathbb{R}$-vector space (exactly in the same way that $\mathbb{R}^3$ is an $\mathbb{R}$-vector space, we just have more coordinates).

   Let R be the set of all $\mathbb{R}$-linear maps from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$. As seen in the previous exercise, R is a ring. Let $f, g \in R$ be the maps defined by
$$g((x_1, x_2, x_3, x_4, \ldots)) = (x_2, x_3, x_4, \ldots) \text{ and}$$
$$f((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, x_3, \ldots).$$
   Show that g is right invertible but not left invertible (hint: write what g left invertible means and show that it would imply g injective) and that f is left invertible but not right invertible (hint: write what f right invertible means and show that it would imply f surjective).