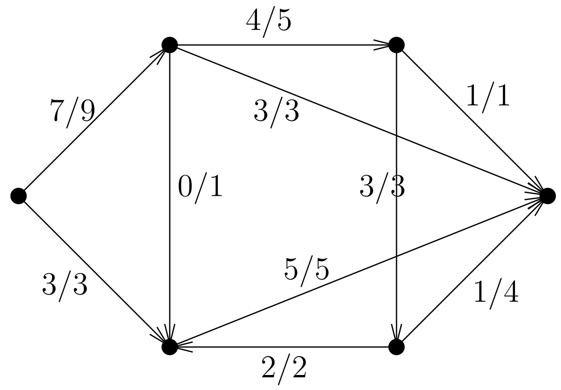
Problem sheet 9

1. Observe first that both the red and blue graphs both contain at most 11 vertices. As seen in class, a planar graph with at most 11 vertices contains at most $3 \times 11 - 6 = 27$ edges. If both the red and blue graphs are planar then K_{11} would contain at most 54 edges, but K_{11} has 55 edges.

2. We simply use the definition of flow and get



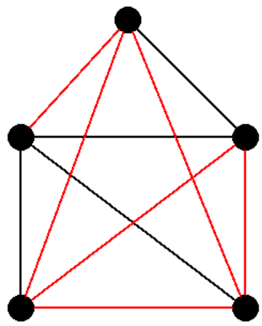
3. Obvious from the definition of the value of a flow.

4. (a) Let $u_1 u_2 \cdots u_k$ be a Hamiltonian cycle in G (so G has $k-1$ vertices and $k-1$ is odd). Say for instance that $u_1 \in X$. Then each u_i with i odd is in X and the others are in Y (since G is bipartite). Since $u_k = u_1$, k is odd and $k-1$ is even, impossible.

   (b) The graph is bipartite (start with one vertex, colour it red, then colour all the other vertices blue and red alternatively. You will end up doing this without having two adjacent vertices of the same colour, so the graph is bipartite) and has an odd number of vertices.

   (c) Build a graph where the vertices are the squares of the chessboard. There are n^2 of them (an odd number since n is odd). Put an edge between 2 vertices if it is possible to go from one to the other by a knight move. It is bipartite since a knight move start and ends at squares of different colours (so the colours give the bipartition). There is no Hamiltonian cycle by the first question.

5. (a) The complement is the red graph. Observe that when you put G and its complement together, you get the complete graph.



(b) Let (X, Y) be the bipartition of the graph. By definition there are no edges between elements of X, so in the complement we will get all possible edges between elements of X. So the elements of X form the vertices of a complete graph. Similarly, the elements of Y form the vertices of a complete graph. And the is no edge in the complement between a vertex in X and a vertex in Y, since there is one in G.