# Problem sheet 8

1. They have the same degree sequences, so nothing obvious there. But they both have two vertices of degree 3, and an isomorphism must send a vertex of degree 3 to a vertex of degree 3 (so there are 2 possibilities for an isomorphism here, with respect to the vertices of degree 3). The vertices of degree 3 in the first graph are adjacent to vertices of degree 1,1,4 and 1,2,2. The vertices of degree 3 in the second graph are adjacent to vertives of degree 1,1,2 and 1,2,4. None of the two possibilities for an isomorphism can do this.

   Other, more clever, possibility: They have both only one vertex of degree 4, so an isomorphism must send one to the other. Then it must send the vertices adjacent to the first one to vertices adjacent to the second one, and also respect the degree. Now, among these vertices, there is one of degree 3 in each graph. The one in the graph on the left has two neighbours of degree 1, and it is not the case with the one on the right, so there is no isomorphism.

   (There are almost certainly other ways to check that they are not isomorphic, and possibly better ways.)

2. (a) Any trivial example with a non-connected planar graph will do.

   (b) Let $k = \omega(G)$ and let $G_i = (V_i, E_i)$ be the components of $G$. Let $F_i$ be the set of faces of $G_i$ and $F_i'$ the set of interior faces of $G_i$. Euler's formula holds for each $G_i$, so $|V_i| - |E_i| + |F_i| = 2$. If we sum all these we get:

   $$\sum_{i=1}^{k} |V_i| - \sum_{i=1}^{k} |E_i| + \sum_{i=1}^{k} |F_i| = 2k,$$

   so

   $$|V| - |E| + \sum_{i=1}^{k} |F_i| = 2k.$$

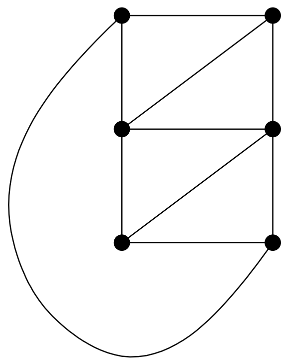
   But $\sum_{i=1}^{k} |F_i| = \sum_{i=1}^{k} |F_i'| + k$. Here we observe that $|F| = \sum_{i=1}^{k} |F_i'| + 1$, so $\sum_{i=1}^{k} |F_i| = |F| + k - 1$. Using this in the formula above we get

   $$|V| - |E| + |F| + k - 1 = 2k,$$

so

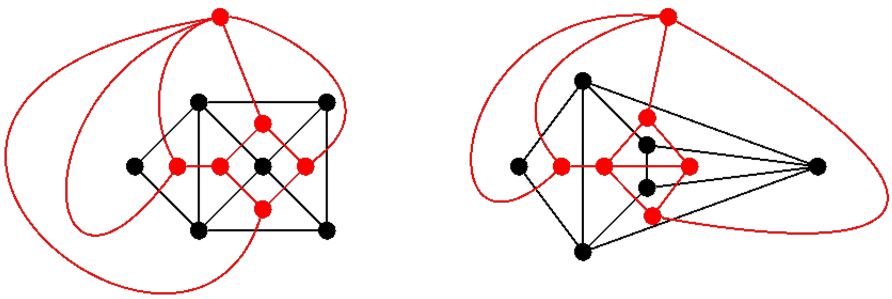
$$|V| - |E| + |F| = \omega(G) + 1.$$

3. (a) For instance



   (b) Just add an edge between any two vertices of $K_{3,3}$.

4. The dual graphs are:



They do not even have the same degree sequence.