Graphs and Networks (MATH20150) 2023/2024

Problem sheet 5

1. Follow the examples seen in class.

2. "$\Rightarrow$" Assume that there is a spanning tree T of G such that e does not belong to T. Then removing e from G does not disconnect G (going from one vertex to another can be done on T, which does not contain e), contradiction.

   "$\Leftarrow$" Assume that e is not a cut edge of G. Then $G \setminus \{e\}$ is connected and thus has a spanning tree T. Then T is a spanning tree of G that does not contain e, contradiction.

3. (a) Let n be the number of vertices of T. We know that the number of edges is $n-1$. The degree sum formula thus gives

   $$\sum_{v\in V} d(v) = 2|E| = 2n-2.$$

   If every vertex had degree at least 2, we would get $\sum_{v\in V} d(v) \geq 2n$, impossible.

   So at least one vertex has degree less than two. Let u be this vertex. Since T has at least 2 vertices, the degree of u is at least 1 (because there is a path from u to some other vertex). Thus the sum of the degrees of the vertices is at least $2n-1$, impossible because it would give $2n-1 \leq 2n-2$.

   Therefore at least 2 vertices must have degree less than 2, i.e. degree 1 (again, since there are at least 2 vertices in T, and a tree is connected, each vertex has degree at least 1).

   (b) Let $P = v_1 \cdots v_k$ be a path of maximal length. Assume that $d(v_k) \geq 2$ (we will show that it is not possible, so that $d(v_k) = 1$, i.e., v_k is a leaf). Then v_k is adjacent to at least v_{k-1} and another vertex u. If u is not one of $v_1, \ldots, v_{k-1}$, then $v_1 \cdots v_k u$ is a path longer than P, which is impossible, so u is one of $v_1, \ldots, v_{k-1}$. Say $u = v_i$. Then $v_i v_{i+1} \cdots v_k u$ is a cyle in T, impossible.

   We show similarly that v_1 is a leaf (it is a correct answer that you can use in an exam, but only if it is really similar).

4. (a) Let $u_1, \ldots, u_k$ be the vertices that v is adjacent to. Let g_i be the component of $G\{v\}$ for $i = 1, \ldots, k$. We first show that $G_1, \ldots, G_k$ are all different (so that $G \setminus \{v\}$ has at least k components.

   Suppose that $G_i = G_j$ with $i \neq j$. Since $u_i, u_j \ni G_i$ (which is connected) there is a path P in G_i from u_i to u_j. But $P' = u_j v u_i$ is a path with edges that are not in P (since these edges are not in $G \setminus \{v\}$. Putting P' at the end of P, we get a cycle in G, impossible.

   We now show that any component of $G \setminus \{v\}$ is equal to one of $G_1, \ldots, G_k$ (so that there are exactly k components). Let H be a component of $G \setminus \{v\}$ and let $u \in H$. Let $P = uv_1 \cdots v_t v$ be a path in G from u to v (observe that $v_1, \ldots, v_t$ are all different from v). Then there is $i \in \{1, \ldots, k\}$ such that $v_t = u_i$ (since v_t is adjacent to v). Therefore $uv_1 \cdots v_t$ is a path in $G \setminus \{v\}$ from u to u_i and thus $u \in G_i$.

   (b) Each G_i (same notation as for the previous question) is a tree (is it connected, and has no cycle since G has no cycle). By Exercise 3, we have 2 cases: G_i has only one vertex, or G_i are at least 2 leaves. In the first case, this vertex is adjacent (in G) to v only, so is a leaf in G. In the second case, at least one of these leaves is not adjacent to v, so is a leaf in G. In both cases, each G_i brings at least one leaf to G. So G has at least k leaves.