Graphs and Networks (MATH20150) 2023/2024

Problem sheet 4

1. This molecule is a tree with $m+n$ vertices. Therefore it has $m+n-1$ edges. Since every C has 4 bonds and every H has one bond, the sum of the degrees is $4m+n$. By the degree sum formula we get that the number of edges is $(4m+n)/2$, i.e. $4m+n = 2(m+n-1)$, so $n = 2m+2$.

2. $A + I_n$ is the adjacency matrix of the graph G' obtained from G by adding a loop at each vertex.

   "$\Rightarrow$" Consider the entry (i,j) of $(A + I_n)^{n-1}$. To show that it is non-zero we must show that there is a path of length $n-1$ in G' from v_i to v_j. Since G is connected, there is a path P in G from v_i to v_j, and this path has length $k \leq n-1$ (since there are only n vertices in G). To get a path of length $n-1$ in G' from v_i to v_j, follow first the path P, then loop $n-1-k$ times at v_j (using the loop at v_j).

   "$\Leftarrow$" Let v_i and v_j be vertices in G. Since $(A + I_n)^{n-1}$ has no zero entries, there is a path of length $n-1$ in G' from v_i to v_j. Obviously we can remove all the loops from this path, and we obtain a path in G from v_i to v_j (we do not know its length).

3. Every time we remove an edge from a cycle the resulting graph is still connected (seen in class). So the graph T is connected. Since it contains no cycles, it is a tree. It still has n vertices (we did not remove any vertex from G), so T has $n-1$ edges. Therefore we removed $m-n+1$ edges.

4. "$\Rightarrow$" If x and y are in the same component of G, since this component is a tree, we know that there is exactly one path from x to y. If x and y are in different components of G, there is no path from x to y.

   "$\Leftarrow$" Let H be a component of G. We show that H is a tree. Let $x, y \in H$. Since H is connected there is at least one path from x to y, and by hypothesis we know that there is exactly one path from x to y. So H is a tree, and G is a forest (since every component is a tree).