Graphs and Networks (MATH20150) 2023/2024

Problem sheet 3

1. It is a bit too tricky to type in detail. The way to do it is to use the Havel-Hakimi algorithm (from the part of the notes entitled "Graphic sequences, adjacency matrix". There are 4 examples in the course notes.

   It works by repeatedly computing the sequence d' out of d. Since the length of the sequence and the numbers in it both decrease, it stops at some point, and it stops either with a sequence with only zeroes (comes from a graph), or a sequence with some negative numbers (does not come from a graph). This is step 1.

   Then to construct the corresponding graph, we follow the proof of "d' graphic $\Rightarrow d$ graphic", so we go "backwards" in the procedure that we used to show that the sequence is graphic. This is step 2.

   The only points to pay attention to are, in step 1: That after each computation of d' it is necessary to put the sequence back into increasing order (it is one of the hypotheses of the theorem). In step 2: After each application of "d' graphic $\Rightarrow d$ graphic", we need to change the order of the vertices to get the degree sequence to correspond to the d' from the previous step. See the examples in the course notes.

   In this exercise, both sequences are graphic.

2. "$\Rightarrow$" Let $C = u_1u_2 \ldots u_n$ be a cycle containing e. We can start numbering the vertices of the cycle such that the first edge is e, i.e. $e = u_1u_2$ (it simplifies the notation). We show that $G \setminus \{e\}$ is connected: Let a, b be vertices in $G \setminus \{e\}$. Since G is connected there is a walk from a to b. If this walk does not contain e, it is still a walk in $G \setminus \{e\}$. If it does contain e it is of the form

   $$av_1 \ldots v_k u_1 u_2 w_1 \ldots w_\ell b.$$

   If we replace $e = u_1u_2$ by the "other side of the cycle" we obtain a walk:

   $$av_1 \ldots v_k u_1 u_n u_{n-1} \ldots u_3 u_2 w_1 \ldots w_\ell b,$$

   from a to b that does not use e. So a is connected to b in $G \setminus \{e\}$.
   "$\Leftarrow$" Write $e = uv$. Since e is not a bridge, $G \setminus \{e\}$ is connected so there is a path

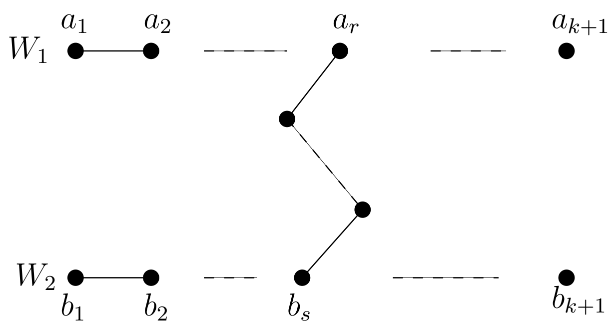
   $$uu_1 \ldots u_k v$$

from u to v in $G \setminus \{e\}$. Therefore $vuu_1 \dots u_k v$ is a cycle in G.

3. Write $W_1 = a_1 a_2 \dots a_{k+1}$ and $W_2 = b_1 b_2 \dots b_{k+1}$ where the a_i and b_j are vertices of G.

   We assume that W_1 and W_2 have no vertex in common. Since G is connected there is a path P in G from the first vertex in W_1 to the first vertex in W_2. This path cannot always be in W_1, otherwise we would have a vertex in both W_1 and W_2. Similarly this path cannot be always in W_2.

   Let a_r be the last vertex in P that is in W_1 and b_s the first vertex in P, that is after a_r and in W_2.



   The length of the path in W_1 from a_1 to a_r is r and from a_r to a_{k+1} is $k - r$ (make a picture with $k = 4$ and $r = 1$). One of them is at least $k/2$. We assume it is the path from a_1 to a_r (the other case is similar).

   Also, the the length of the path in W_2 from b_1 to b_s is s and from b_s to b_{k+1} is $k - s$. Again one of them is of length at least $k/2$, let us assume it is the path from b_1 to b_s (again, the other case will be similar).

   We now build a new path in G as follows: We follow W_1 from a_1 to a_r (this part of the path has length $r \geq k/2$), then P from a_r to b_s (observe that by choice of a_s and b_s there are no elements of W_1 or W_2 in this partion of P; this part of the path has length at least one since $a_r \neq b_s$)) and finally we follow W_2 from b_s to b_1 (this part of the path has length $r \geq k/2$). This new path has therefore length greater than $k/2 + k/2 = k$, which is a contradiction.

4. No solution, just base your drawings on Euler's description of Eulerian graphs.