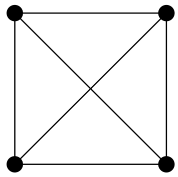
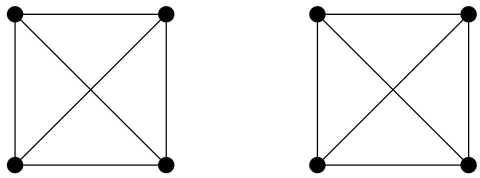
Problem sheet 2

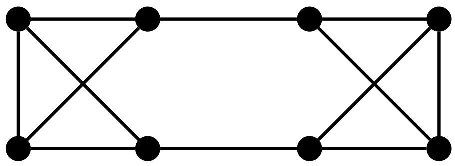
1. We start by figuring out what the smallest 3-regular graph could be: If we have one vertex, it needs to have degree 3, so needs to be adjacent to 3 other vertices. So we must have at least 4 vertices, and in this case there is only one possibility for a 3-regular graph:



   Based on this, we can easily construct a disconnected 3-regular graph:



   Now, we try to construct a connected one with the same vertices. A bit a trial and error should give an answer. One possibility is:



2. (a) The points on the circle must be at distance at least one to each other. If we want to maximise the number of points on the circle, we need to take them at distance exactly one to each other. If means that if u, v are two points on the circle such that the distance from u to v is one, the triangle xuv is equilateral, so the angle at x is $\pi/3$. Since $2\pi = 6 \times \pi/3$, we can have at most 6 points on the circle

(a) Let x be a vertex. The vertices adjacent to x are on the circle of centre x and radius 1 and at distance at least one to each other. By the previous question there are at most 6 of them.

(b) By the degree sum formula we have $\sum_{x \in V} d(x) = 2|E|$. Using the previous question we obtain $2|E| \leq 6n$, so $|E| \leq 3n$.

3. Recall that a path is a walk with no repeated vertices. So, since there are only finitely many vertices in the graph, there can be only finitely many paths. Therefore (at least) one of them has maximal length.

   Suppose $r \leq k$. By hypothesis we know that v_r is connected to at least k vertices. At least one of these (call it w) is therefore not in the list $v_1, \ldots, v_{k-1}$. So the $v_1 v_2 \cdots v_r w$ is a path that is longer than W, contradiction.

4. (a) We can use the Havel-Hakimi result. Applying is successively, we get the following sequences:

$$(1, 1, 3, 3, 5, 5), (0, 0, 2, 2, 4), (-1, -1, 1, 1).$$

(Observe that in this example there is no need to put each sequence back in non-decreasing order. It is just luck: in general the new sequence you obtain may not be in increasing order, and you need to re-order it to continue applying the Havel-Hakimi result.)

The third sequence is not graphic (it contains negative numbers), so the original sequence is not graphic.

(b) Applying again the Havel-Hakimi result, we successively get the following sequences:

$$(1, 2, 2, 3, 4), (0, 1, 1, 2), (0, 0, 0).$$

The final sequence is graphic, so the original one is also graphic.

Remark: If you directly produce a graph with the correct degree sequence, it is of course a proof that the sequence is graphic. It can sometimes be done for short or simple sequences.