Graphs and Networks (MATH20150) 2023/2024

Problem sheet 3

1. Determine which of the following sequences are graphic, and, for each graphic sequence, find a graph with this sequence.
   (a) $(2, 3, 3, 4, 4)$.
   (b) $(1, 1, 2, 3, 3, 5, 5, 6)$.

2. Let e be an edge in a connected graph G. Recall that e is called a bridge (also called a cut-edge) if $G \setminus \{e\}$ is not connected.
   (a) Show that if e is contained in a cycle, then e is not a bridge.
   (b) Show that if e is not a bridge, then e is contained in a cycle. Hint: If $e = uv$ with u, v vertices, is there a path from u to v? In which graph?

3. Let G be a connected graph and let k be the maximal length of a path in G. Let W_1 and W_2 be two paths of length k in G. Show that W_1 and W_2 have a vertex in common.
   Hint: Assume that it is not the case. Show that there is a path of length at least one from a vertex in W_1 to a vertex in W_2. Use this to construct of path of length greater than k.

4. Draw an example of an Eulerian graph, and of a graph that is not Eulerian (make sure that your graphs are not too simple).

The following is simply the proof of Euler's theorem, presented as an exercise. No solution will be given in the tutorials (look at your course notes). It is there to point out that the proofs in class are not very removed from being exercises, and maybe also to give you some idea about how to work on the proofs we see.

5. Let G be a connected graph such that every vertex in G has even degree. Let W be a walk in G without repeated edges and of maximal length with this property. We write

$$W = v_0 e_0 v_1 e_1 \cdots v_{k-1} e_{k-1} v_k,$$

   where the v_i are vertices and the e_i edges. Answering the following questions will show that W is an Euler circuit in G.

(a) Show that every edge adjacent to v_k must appear as an edge in W (consider what happens if it is not the case; it relies on the properties of W).

(b) Show that $v_0 = v_k$. Hint: Consider how the degree of v_k is linked to the number of times v_k appears in W (don't forget that the vertices in W are not necessarilly all different, so that v_k can appear several times).

(c) Show that if some edge of G is not in W, then there is an edge of G that is not in W but is adjacent to a vertex of W.

(d) Deduce that every edge of G is in W (use the property of W to reach a contradiction if it is not the case).