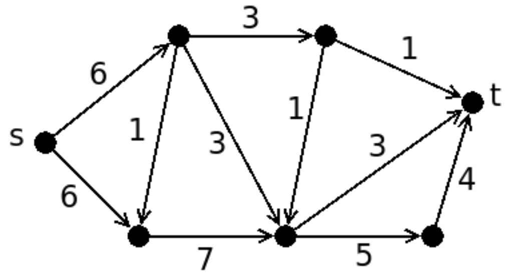
Problem sheet 10

1. Find a maximal flow and a minimal cut in the following network (the source is s, the sink it t and the numbers indicate the capacities of the arcs).



2. Let f be a flow on a network N with capacity c. Are the following statements true? (For each of them: Prove it or produce a counter-example.)
   (a) f is maximum implies that for every arc e, $f(e) = 0$ or $f(e) = c(e)$.
   (b) There is a maximal flow f such that, for every arc e, $f(e) = 0$ or $f(e) = c(e)$.
   (c) Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Multiplying all capacities by α does not change the minimal cuts.
   (d) Let $\alpha \in \mathbb{R}$, $\alpha > 0$. Adding α to all capacities does not change the minimal cuts.

3. Not an exercise, just an example to show that networks and flows can be used to determine if some goods can be delivered in the righ quantity: Assume you have a set of factories $F_1, \dots, F_n$ producing some goods and a set of towns $T_1, \dots, T_k$ where they have to be delivered. You have arcs from the factories to the places, with capacity indicating how much can be transported from one to the other. To model that a factory F_i can produce at most m_i, you add a single source s and an arc from s to F_i with capacity m_i. To model the fact that a town T_i demands a quantity q_i, add a single sink t and an arc of capacity q_i from T_i to t. It is then possible to satisfy the demand if there is a flow of capacity $q_i + \cdots + q_k$. It will be a maximal flow (why?), and can be found with the Ford-Fulkerson algorithm.

4. The thickness $t(G)$ of a graph G is the smallest number of planar graphs (with the same vertices as G and disjoint sets of edges) that can be superimposed to form G. For instance the thickness of a planar graph is 1, the thickness of K_5 is 2.

Show that if $G = (V, E)$ is a graph, then

$$t(G) \geq \frac{|E|}{3|V| - 6}.$$

Hint: Write $G = G_1 \cup \cdots \cup G_t$ with $G_1, \ldots, G_t$ planar.

5. Let G be a directed graph with set of vertices V. G is called reducible if there are two subsets non-empty V_1 and V_2 of V such that

   (a) $V = V_1 \cup V_2$,

   (b) $V_1 \cap V_2 = \emptyset$,

   (c) Every arc between a vertex in V_1 and a vertex in V_2 goes from V_1 to V_2 (i.e., there is no arc going from V_2 to V_1).

   G is called irreducible if it not reducible.

   G is called strongly connected if for every $u, v \in V$ there is a directed path in G from u to v (i.e., G is connected using directed paths).

   (a) Assume that G is irreducible and let $u \in V$. Let $V(u)$ be the set of all vertices that can be reached from u by a directed path. Show that $V(u) = V$. Hint: Look at $V(u)$ as the set V_1 above.

   (b) Show that G is irreducible if and only if G is strongly connected.