MATH20150 - 2019 exam - Solution

1. (a) Give it the weight 2.

   (b) Yes: put the long vertical edge starting in v to the left side.

2. (a) Yes: If you put a vertex at each intersection of lines, and count the lines as edges, we have an Eulerian graph (the degree of each vertex is even).

   (b) Let $v_1, \ldots, v_n$ be the vertices of G. The i-th diagonal entry of A^2 is the number of walks of length 2 from v_i to v_i (seen in class), so is the degree of v_i. The result follows since the sum of the degrees of the vertices is $2|E|$.

3. (a) i. $\Rightarrow$ ii. Let e be an edge in the cycle of G. Then $G \setminus \{e\}$ has no cycle, and is still connected. So $G \setminus \{e\}$ is a tree.

   11. $\Rightarrow$ i. Because adding a new edge to a tree adds a single sycle (seen in class).

   (b) Let G_1, G_2 be two components of $G \setminus S$. Take a path in T from a vertex in G_1 to a vertex in G_2. At least one edge of this path in not in $G \setminus S$ since G_1 and G_2 are different components. Since this edge is not in $G \setminus S$, it is in S.

4. (a) Easy, just follow the definition. Note that each vertex of this line graph has even degree (4 to be precise).

   (b) Let $e = uv$ be an edge in G / a vertex in $L(G)$. Say that $d(u) = 2r$ and $d(v) = 2s$. Then, in $L(G)$, e is adjacent to $2r - 1$ vertices thanks to u (the other edges that u is adjacent to), and to $2s - 1$ vertices thanks to v. So $d_{L(G)}(e) = 2(r + s) - 2$ which is even. Every vertex of $L(G)$ has even degree, so $L(G)$ is Eulerian.

   (c) No: A counter-example is provided by K_4 (not Eulerian), cf. first question. It is also easy to come up with another graph with 4 vertices where it does not work.

5. (a) Let v be the number of vertices, e the numbre of edges, and f the number of faces. Since $v - e + f = 2$ we get that $f = 8$. We also know that a face has degree at least 3, so if s is the sum of the degrees of the faces, we have $s \geq 24$, with equality when each face has degree 3. But we know that $s = 2e = 24$, so each face has degree 3.

   (b) Follow the algorithm seen in class: It will give a maximum flow (of value 10), and can also be used to get a minimal cut (or: the minimal cut can be found by inspection, the graph is small enough). The fact that the cut is minimal comes either from finding it via the algorithm, or using the max-cut / min-flow theorem (check that its capacity is equal to the value of the maximum flow).