From Numerical Analysis to Analysis of the Numerics

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Outline

- Where I am coming from
- The Mixed Finite Element
  - The *p*-version
  - The Moving Meshes
- Fracture in The Ice
- Heat Problem
- NASA Work
- The Study of the Bay
- Biofuels Studies

Where I am coming from

Where I am coming from

Figure: U.S.Naval Academy

The Mixed Finite Element

The Quasilinear Parabolic Equation

$$c(x,u)\frac{\partial u}{\partial t} - \nabla(a(x,u)\cdot\nabla u) = f(x,u,t), \quad x \in \Omega,\ t \in \mathbf{J},$$

$$u(x,0) = u_0, \quad x \in \Omega,\ t = 0,$$

$$u(x,t) = -g(x,t), \quad x \in \partial\Omega,\ t \in \mathbf{J}$$

where Ω is a bounded domain in $\mathcal{R}^2$, its boundary $\partial\Omega$ is C^∞, and $\mathbf{J}= (0, T]$.

Let

$$H(\text{div},\Omega) := \left\{\chi \in (L_2(\Omega))^2; \text{div}\chi \in L_2(\Omega)\right\}$$

and the norm for $H(\text{div},\Omega)$ by

$$||\chi||_H^2 = ||\chi||_0^2 + ||\text{div}\chi||_0^2$$

Formulation of the Mixed Finite Element Method

The flux variable $\sigma = -a(x,u)\nabla u,$ and $\alpha(u) := \alpha(x,u) = \dfrac{1}{a(x,u)}.$

Saddle-Point Problem

$$(u,\sigma) \in L_2 \times H$$

$$(\alpha(u)\sigma, \chi) - (\mathrm{div}\chi, u) = < g, \chi \cdot \nu >, \quad \chi \in H,$$

$$(c(u)u_t, \nu) + (\mathrm{div}\sigma, \nu) = (f(u), \nu), \quad \nu \in L_2.$$

$$u(x,0) = u_0,$$

where $(\cdot,\cdot)$ is the usual $L_2(\Omega)$ inner product, $< \cdot,\cdot >$ is the L_2 inner product on the boundary of Ω given by $< w, v >= \int_{\partial\Omega} w\, v\; ds$, and v is the unit outward normal vector to $\partial\Omega$.

Main Result

Saddle-Point Problem

$$(u_h, \sigma_h) \in V_h \times H_h$$

$$(\alpha(u_h)\sigma_h, \chi_h) - (\mathrm{div}\chi_h, u_h) =< g, \chi_h \cdot \nu_h >, \quad \chi_h \in H_h,$$

$$(c(u_h)u_{ht}, \nu_h) + (\mathrm{div}\sigma_h, \nu_h) = (f(u_h), \nu_h), \quad \nu_h \in V_h.$$

$$u_h(x, 0) = U(0),$$

where $U(0)$ is the elliptic mixed method projection into the finite-dimensional space V_h of the initial data function $u(0)$, and $V_h \subset L_2$ and $H_h \subset H(\mathrm{div}, \Omega)$.

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Theorem

There is a constant C, independent of h, such that

$$||u - u_h||_{L_\infty(J;L_2)} + ||\sigma - \sigma_h||_{L_\infty(J;H(\text{div}))}$$

$$\leq C \left\{ ||u||_{L_2(J;H^{r+1+\delta_{k0}}(\Omega)} + ||u_t||_{L_2(J;H^{r+1+\delta_{k0}}(\Omega)} \right\} h^r, \quad 0 \leq k < r + 1,$$

For Further Reading

Improved error estimates for mixed finite-element approximations for nonlinear parabolic equations: The continuous-time case.
Numer. Methods Partial Differ. Equations, 10(2):129-147, 1994.

Improved error estimates for mixed finite-element approximations for nonlinear parabolic equations: The discrete-time case.
Numer. Methods Partial Differ. Equations, 10(2):149-169, 1994.

- With Soren Jensen
  The p-version of mixed finite element methods for parabolic problems.
  RAIRO. Modelisation Mathematique et Analyse Numerique, 31(3):303-326, 1997.

- With Yingjie Liu, Randolph E. Bank, Todd F. Dupont and Rafael F. Santos
  Symmetric Error Estimates for Moving Mesh Mixed Methods for Advection-Diffusion Equations.
  SIAM Journal on Numerical Analysis, 40(6):2270-2291, 2002.

Fracture in The Ice

Figure: http://www.aad.gov.au/

Problem Modeling Shear Band Formation

This problem consists of a system of two coupled partial differential equations with velocity and temperature unknowns.
Find $v = v(x,t)$ and $\theta = \theta(x,t)$ such that

Modeling Shear Band Formation

$$\dot{v} - \nabla \cdot (\mu(\theta)\nabla v) = f, \quad \text{in } \Omega \times [0, T], \tag{1}$$

$$\dot{\theta} - \Delta\theta - \mu(\theta)|\nabla v|^2 = g, \quad \text{in } \Omega \times [0, T], \tag{2}$$

$$v = \theta = 0, \quad \text{on } \Gamma \times [0, T], \tag{3}$$

$$v(\cdot, 0) = v_0 \text{ and } \theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega, \tag{4}$$

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where Ω is a bounded domain in $\mathbf{R}^d, d = 1, 2$ with smooth boundary Γ and $\dot{w} = \partial w/\partial t$. The functions v and θ are real valued, T is a fixed positive constant and f, g, v_0 and θ_0 are given data.

Modeling Shear Band Formation

Case where $\mu(\theta) = e^{-\alpha\theta}$ for $\alpha > 0$.

The quantity $\sigma = \mu(\theta)\nabla v$ is the stress.

Find $v_h : [0, T_h] \to \mathcal{M}_h$ and $\theta_h : [0, T_h] \to \mathcal{M}_h$ such that

Variational Equations

$$(\dot{v}_h, \chi) + (\mu(\theta)\nabla v_h, \nabla\chi) = (f, \chi), \quad \forall\chi \in \mathcal{M}_h, \tag{5}$$

$$(\dot{\theta}_h, \lambda) + (\nabla\theta_h, \nabla\lambda) - (\mu(\theta)|\nabla v_h|^2, \lambda) = (g, \lambda), \quad \forall\lambda \in \mathcal{M}_h, \tag{6}$$

where

$$v_h(\cdot, 0) = v_{0h}, \quad \theta_h(\cdot, 0) = \theta_{0h}, \text{ in } \Omega, \tag{7}$$

Modeling Shear Band Formation

The functions $v_{0h} \in \mathcal{M}_h$ and $\theta_{0h} \in \mathcal{M}_h$ are given approximations to v_0 and θ_0, respectively.

After introducing a basis for $\mathcal{M}_h$, this system becomes a system of nonlinear ordinary differential equations.

Theorem

Under the hypotheses above v_h and θ_h are solutions of the system of non-linear ordinary differential equations on $[0, T]$ and there exist constants $C = C(v, \theta, T)$ and $h_0 = h_0(v, \theta, T)$ such that

$$||(v - v_h)(\cdot, t)||_0 + ||(\theta - \theta_h)(\cdot, t)||_0 \leq \begin{cases} Ch^{2-\epsilon} & \text{if } p = 1, \\ Ch^{p+1} & \text{if } p \geq 2, \end{cases} \quad (8)$$

for $0 \leq h \leq h_0$, $0 \leq t \leq T$ and ϵ small.

For Further Reading

With Donald A. French
Finite element approximation of an evolution problem modeling shear band formation.
Comput. Methods Appl. Mech. Eng., 118(1-2):153-161, 1994.

Heat Problem

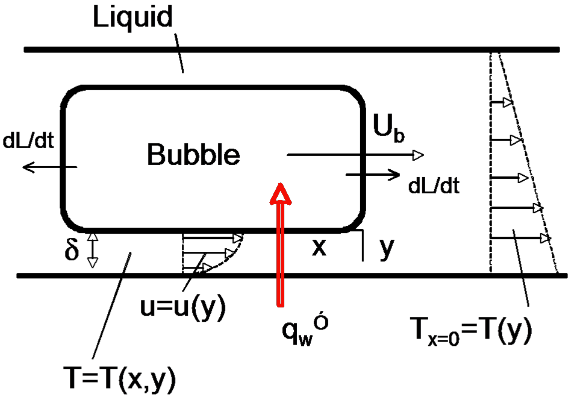


Figure: 2-D cross-section of single bubble in channel, for large z/δ

Heat Problem

The model for bubble growth rate will be obtained using the following relationship which is heat transfer dominant and inertia effects have been neglected in this first approach:

$$\rho_g h_{fg} \frac{\partial(Vol)}{\partial t} = k \int_0^L \left(\frac{\partial T}{\partial y}\right)_{y=0} dA$$

Inside the bubble, the following provides the standard definition for the local heat transfer coefficient:

$$h_x = \frac{k \left(\frac{\partial T}{\partial y}\Big|_{y=0} \right)}{T_{\text{wall}} - T_{\text{sat}}}$$

Heat Problem

The following equation has been developed with respect to an origin fixed at the bubble leading edge:

Formulation

$$\alpha \frac{\partial^2 T}{\partial y^2} = \left\{ U_b + \frac{dL}{dt} - u(y) \right\} \cdot \frac{\partial T}{\partial x}$$

where U_b is the bubble velocity through microchannel, L is the length of the bubble along the x-axis, T is the temperature (K) in liquid layer under bubble base, T_{sat} is the coolant saturation temperature, and T_{wall} is the temperature at channel wall.

For Further Reading

- With Martin Cerza.
  An Analytical Model For Multiple Bubbles Growth And Heat Transfer In Micro-Channels.
  Work in Progress

- With Martin Cerza and Josh L. Nickerson.
  An Analytical Model For Bubble Growth And Heat Transfer In Micro-Channels.
  Proceedings of IMECE2007-43994 ASME International Mechanical Engineering Congress and Exposition, November 11-15, 2007, Seattle, WA, USA.

- With Martin Cerza and Josh L. Nickerson.
  Bubble Growth Heat Transfer In Micro-Channel Cooling Systems.
  Proceedings of IMECE2007 ASME International Mechanical Engineering Congress and Exposition, November 11-15, 2007, Seattle, WA, USA.

NASA Work

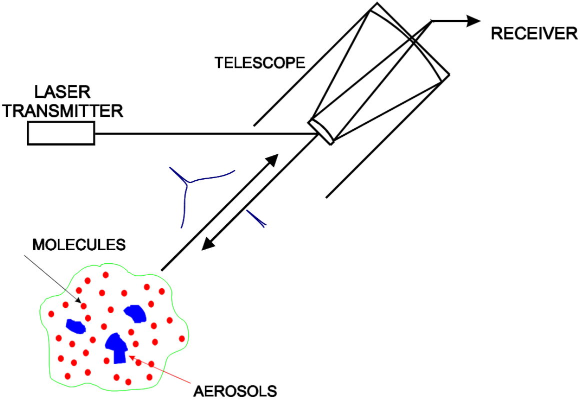


Figure: LIDAR **LI**ght **D**etection **A**nd **R**anging

NASA Work

Figure: Holographic Airborne Rotating Lidar Instrument Experiment, or **HARLIE**

NASA Work

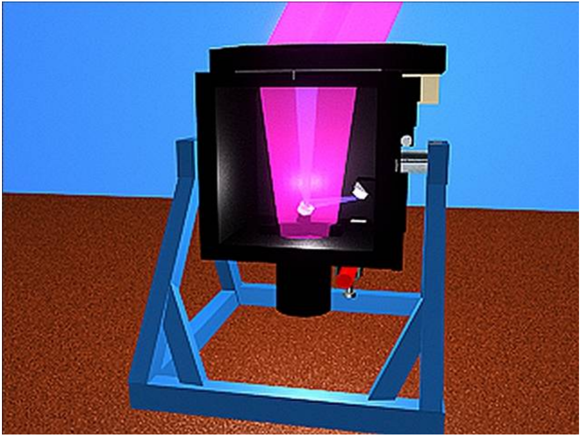
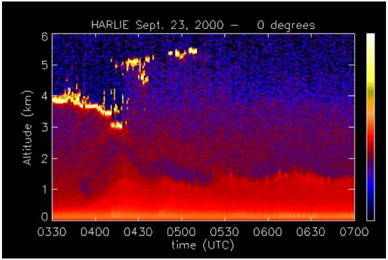


Figure: Holographic Airborne Rotating Lidar Instrument

NASA Work

Data Products

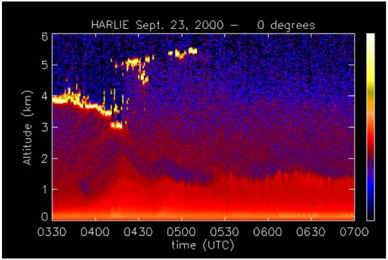
- Real-Time Aerosol Backscatter Profiles



NASA Work

Data Products

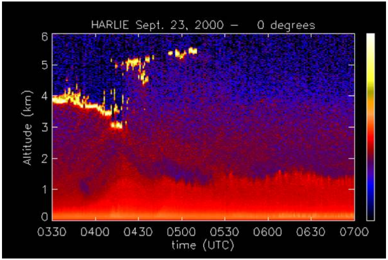
- Real-Time Aerosol Backscatter Profiles
- Boundary Layer top



NASA Work

Data Products

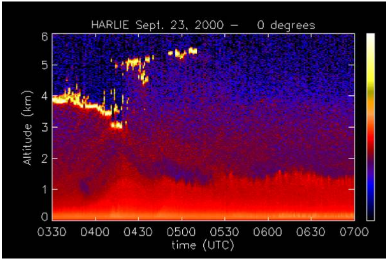
- Real-Time Aerosol Backscatter Profiles
- Boundary Layer top
- Entrainment Zone Thickness



NASA Work

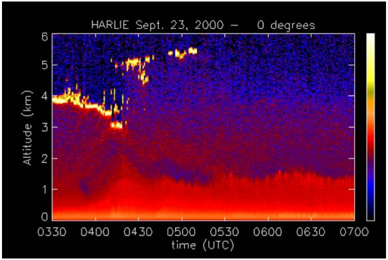
Data Products

- Real-Time Aerosol Backscatter Profiles
- Boundary Layer top
- Entrainment Zone Thickness
- Cloud Bottom Heights



Data Products

- Real-Time Aerosol Backscatter Profiles
- Boundary Layer top
- Entrainment Zone Thickness
- Cloud Bottom Heights
- Wind Profiles



NASA Work - Contributing Investigators

- Geary Schwemmer, NASA GSFC
- David Miller, SSAI
- David Whiteman, Bruce Gentry, NASA GSFC
- David Miller, Sangwoo Lee, Stephen Palm, SSAI
- Thomas Wilkerson, Ionio Andrus, Cameron Egbert, Mark Anderson, Jason Sanders, Utah State University
- David Guerra, Saint Anselm College
- Belay Demoz, Keith Evans, UMBC

The Study of the Bay

Figure: CCPOM Project

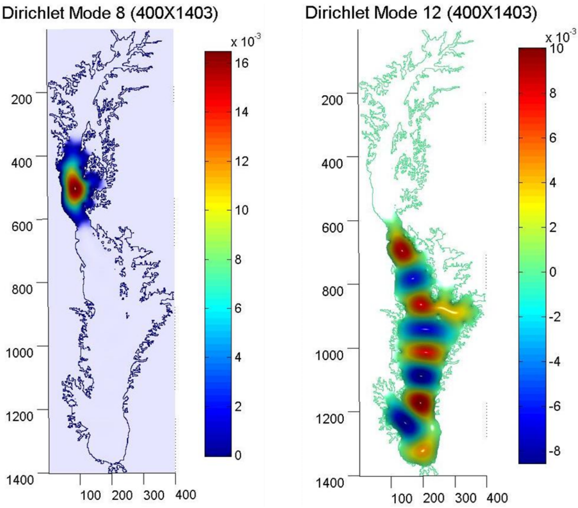


Figure: CCPOM Project

Biofuels Studies

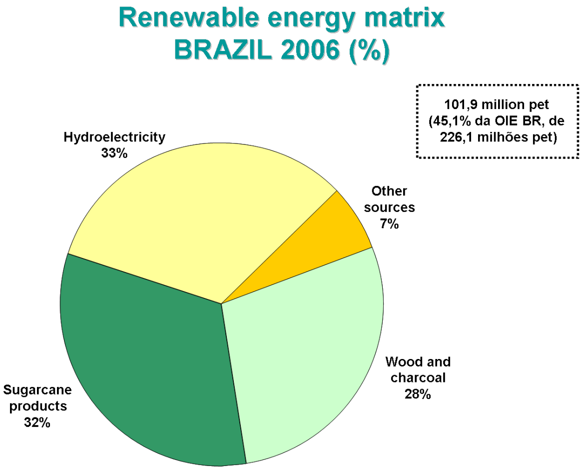
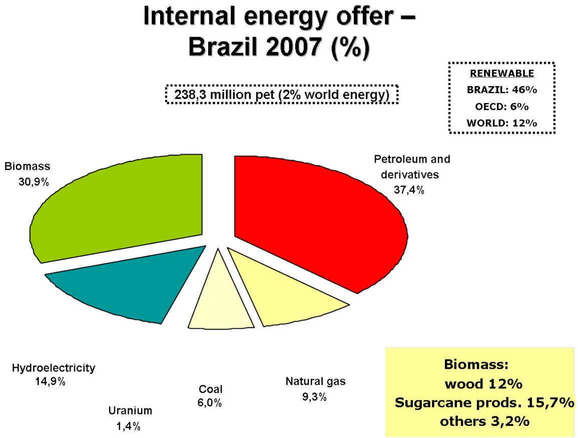


Figure: From Dr. Paulo Cezar Vieira, UFSCar, Brazil

Biofuels Studies



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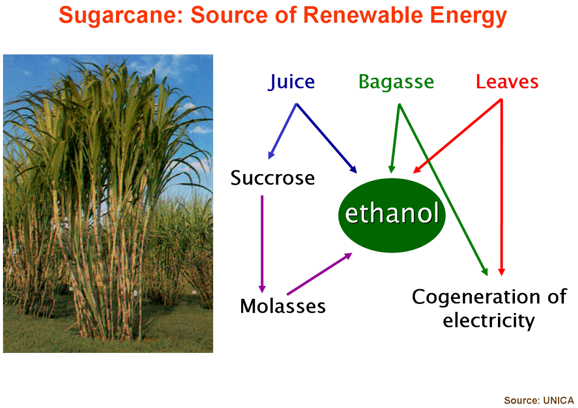
Biofuels Studies



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