UNIVERSITY COLLEGE DUBLIN

NATIONAL UNIVERSITY OF IRELAND, DUBLIN

An Coláiste Ollscoile Baile Átha Cliath

Ollscoil na hÉireann, Baile Átha Cliath

SUMMER EXAMINATIONS 2005

SCMXF0028
SCMXP0028

MATHEMATICAL PHYSICS

Numerical Weather Prediction

MAPH-P313

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Instructions for Candidates

Please use *separate answer book* for each question
All questions carry equal marks
Full marks for complete answers to *four* questions.

Time: 3 hours.

Notes for Invigilators
Non-programmable calculators are permitted

1. (a) Define what is meant by the characteristics of the linear advection equation for $u(x,t)$,

$$\frac{\partial u}{\partial t} + v(x,t)\frac{\partial u}{\partial x} = 0$$

where $v(x,t)$ is a known function and show that $u(x,t)$ is constant along such characteristics. Derive the upwind scheme for the linear advection equation on a rectangular grid on the (x,t) plane. Sketch the characteristics of the equation $u_t + v(x)u_x = 0$ for $0 \leq x \leq 1$ and $v(x) = x - \frac{1}{2}$. Explain why no boundary conditions are needed. Using the characteristics, find the solution when $u(x,0) = x(1-x)$.

(b) Construct the upwind scheme for the linear advection equation on a rectangular grid. What is the truncation error of the upwind scheme ?

(c) Construct the upwind scheme for the inviscid Burger's equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

Verify that the solution to Burger's equation with initial condition $u(x,0) = u_0(x)$ is given implicitly by $u(x,t) = u_0(x - u(x,t)t)$.

2. (a) The Von Neumann stability analysis for a finite difference scheme consists of inserting the trial solution $u_j^n = \xi^n e^{ik(jh)}$ for the numerical value of u at the j^{th} spatial and n^{th} temporal point, where k is the wavenumber, h the spatial stepsize and ξ the amplitude factor. Describe why $|\xi| < 1$ is a necessary stability constraint for a difference solution of the linear advection equation.

(b) Construct the forward-time, centred-space (FTCS) method for the linear advection equation and show that is is unconditionally unstable. Modify that method to give the Lax technique and show that the amplification factor is given by

$$\xi = \cos(kh) - i\frac{v\tau}{h}\sin(kh)$$

where τ is the timestep. Use this result to define the Courant-Friedrichs-Lewy (CFL) stability constraint and numerical damping of the solution.

(c) Construct the leapfrog scheme for the linear advection equation and show that there is no damping when the CFL condition is satisfied.

3. (a) An approximation to the two-dimensional linear advective equation is given in standard notation by

$$\frac{\partial u}{\partial t} = -c_x\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - c_y\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}$$

where $x = i\Delta x, y = j\Delta y$ and c_x, c_y are the *constant* components of the advective velocity. If a leapfrog time differencing scheme is employed with timestep Δt show that we obtain the stability criterion

$$\sqrt{2}\,\frac{c\Delta t}{d} \leq 1$$

where $c_x = c_y = c/\sqrt{2}$ and $\Delta x = \Delta y = d$. How does this differ from the one-dimensional case?

(b) The linearised shallow-water equations with rotation, for velocities u, v and height h independent of y, are approximated on the Arakawa C grid by

$$\begin{aligned}\frac{\partial u}{\partial t} &= -g\delta_x h + f\overline{v^x} \\ \frac{\partial v}{\partial t} &= -f\overline{u^x} \\ \frac{\partial h}{\partial t} &= -H\delta_x u\end{aligned}$$

where H is the mean depth, with the difference and smoothing operators

$$(\delta_x a)_{i,j} = \frac{1}{d}\left(a_{i+\frac{1}{2},j} - a_{i-\frac{1}{2},j}\right)$$

$$\left(\overline{a^x}\right)_{i,j} = \frac{1}{2}\left(a_{i+\frac{1}{2},j} + a_{i-\frac{1}{2},j}\right)$$

Derive the dispersion relationship

$$\left(\frac{\omega}{f}\right)^2 = \cos^2\frac{kd}{2} + 4\left(\frac{gH}{f^2d^2}\right)\sin^2\frac{kd}{2}$$

for waves of frequency ω and wavenumber k.

4. (a) A dynamic forecast model based on the barotropic vorticity equation requires the solution of the Poisson equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta$$

for the stream function ψ, given the vorticity ζ.
Show that $\partial^2\psi/\partial x^2$ can be approximated to $O(\Delta x^2)$ by the finite difference

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2}$$

where $\psi_{i,j} = \psi(i\Delta x, j\Delta y)$ for a mesh of size $\Delta x, \Delta y$.
Assume a domain $0 \le x \le 1, 0 \le y \le 1$ and a square mesh of size $1/4$. Set up the matrix equation satisfied by a finite-difference approximation for $\psi(x,y)$. You should assume that ψ is zero on the boundary of the domain.

(b) Construct the Jacobi algorithm for the iterative solution of the matrix equation

$$\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$$

Show how the algorithm is modified to give the Gauss-Seidel method.

(c) Carry out four iterations of the Gauss-Seidel method for the system of equations

$$\begin{aligned} 4x_1 - x_2 - x_3 &= 1 \\ -x_1 + 4x_2 - x_4 &= 2 \\ -x_1 + 4x_3 - x_4 &= 0 \\ -x_2 - x_3 + 4x_4 &= 1 \end{aligned}$$

Use $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$ as the initial estimate.

5. (a) Two sets of independent measurements T_1, T_2 are taken to determine the scalar T.

i. Outline the least squares method to determine the estimate $T_a = a_1 T_1 + a_2 T_2$. Hence, find the weights a_1, a_2 in terms of the variances σ_1^2, σ_2^2 of the measurements. Show that the precision of the analysis is the sum of the precision of the measurements.

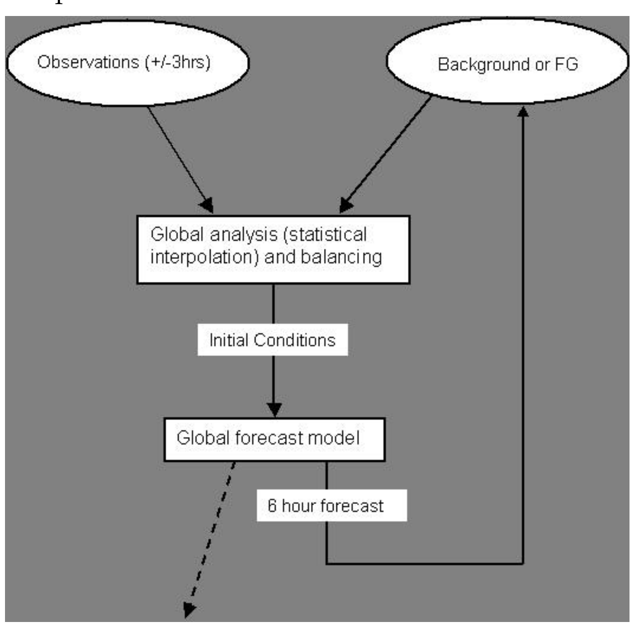
ii. The likelihood of a true value T given an observation T_o with a standard deviation σ_o is given by

$$L_{\sigma_o}(T||T_o) = p_{\sigma_o}(T_o|T) = \frac{1}{\sqrt{2\pi}\sigma_o} \mathrm{e}^{\frac{-(T_o - T)^2}{2\sigma_o^2}}$$

Show that the maximum likelihood value of T is given by the minimum of the cost function

$$J(T) = \frac{1}{2}\left[\frac{(T - T_1)^2}{\sigma_1^2} + \frac{(T - T_2)^2}{\sigma_2^2}\right]$$

(b) Provide a brief commentary on the 6-hour analysis cycle as illustrated below. Include a description of the successive correction method of analysis.



Operational Forecasts

6. (a) Consider the 1D advection equation $d\psi/dt = 0$, where d/dt is the material derivative, for the case of a constant wind u (assume $u > 0$). The integral of this equation along the trajectory of a particle that arrives at the gridpoint $x_I (= I\Delta x)$ at time $(n+1)\Delta t$, having departed from the point x_* at time $n\Delta t$, is given by

$$\psi_I^{n+1} = \psi_*^n \tag{1}$$

Write down the semi-Lagrangian integration scheme based on (1) that uses quadratic interpolation of the gridpoint values of ψ from the three grid points nearest to x_* (the central point of the three being chosen as the point to which x_* lies closest). Show that for such a scheme the amplification factor for the Fourier component of wavenumber k is

$$A_k = \Big[1 - \hat{\alpha}^2(1 - \cos k\Delta x) - i\hat{\alpha}\sin k\Delta x\Big]\exp(-ipk\Delta x) \tag{2}$$

where $\alpha = u\Delta t/\Delta x$, p is the closest integer to α, and $\hat{\alpha} = \alpha - p$.

(b) Using (2), show that the scheme in question is unconditionally stable.

(c) Discuss the implications of the semi-Lagrangian approach for NWP.

7. (a) Consider a climate model consisting of a single isothermal atmospheric layer of temperature T_A which is transparent to solar radiation and of emissivity ϵ_A for longwave radiation, overlying a surface which is of emissivity 1 for longwave radiation. Let S be the mean solar radiation absorbed by the surface per unit area and let H be the turbulent surface flux per unit area (positive upwards).

Show that the model's equilibrium climate is characterized by

$$E_S = \frac{S - H/2}{1 - \epsilon_A/2}$$

where E_S is the surface longwave emission.

(b) Calculate the planetary emission temperature (T_e), the surface temperature (T_S) and the atmospheric temperature (T_A) for the above model when $S = 240\,\mathrm{W\,m^{-2}}$, $H = 0$ and $\epsilon_A = 0.9$.

(c) What value of H is required to reduce the surface temperature to the observed value of 288 K? What is the atmospheric temperature in this case?

(d) Using the definition of the greenhouse effect G in terms of temperatures, what is the value of G in (b) and (c) above?

Note: The Stefan-Boltzmann constant σ is given by $5.67 \times 10^{-8}\,\mathrm{W\,m^{-2}K^{-4}}$.

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