University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER I EXAMINATION 2008/2009

MAPH 40230

Dynamic Meteorology

Extern examiner: Prof Keith Shine

Head of School: Prof Séan Dineen

Examiner: Prof Ray Bates*

Time Allowed: 3 hours

Instructions for Candidates

Full marks will be awarded for complete answers
to **four** questions.
All questions carry equal marks.
Please do not use red pen on the answer books.
Please use separate answer book for each question.

Instructions for Invigilators

Non-programmable calculators may be used during this examination.

Question 1.

(a) Define circulation and (3D) vorticity in an absolute frame of reference.

(b) For a fluid governed by the inviscid 3D momentum equation in an absolute frame, i.e.,

$$\frac{D_a\vec{U}_a}{Dt} = -\frac{1}{\rho}\vec{\nabla}p - \vec{\nabla}\Phi \qquad (1.1)$$

where $-\vec{\nabla}\Phi$ is the gravitational force and the remaining notation is conventional, show that the total derivative of the absolute circulation satisfies the following relationship

$$\frac{D_a C_a}{Dt} = -\oint\frac{dp}{\rho} \qquad (1.2)$$

(c) Show that if the fluid is barotropic eq. (1.2) reduces to

$$\frac{D_a C_a}{Dt} = 0 \qquad (1.3)$$

(Kelvin's circulation theorem).

Question 2.

(a) Explain the difference between a body force and a surface force acting on a particle of fluid and mention an example of each. Also, explain the difference between a real (or true) force and an apparent (or fictitious) force and mention an example of each.

(b) The total derivative of an arbitrary vector $\vec{A}$ in an inertial frame $(D_a\vec{A}/Dt)$ is related to that in a rotating frame $(D\vec{A}/Dt)$ by the expression

$$\frac{D_a\vec{A}}{Dt} = \frac{D\vec{A}}{Dt} + \vec{\Omega}\times\vec{A} \qquad (2.1)$$

where $\vec{\Omega}$ is the rotation vector. Using this expression, show that Newton's second law of motion in the rotating frame can be written

$$\frac{D\vec{U}}{Dt} = \sum\vec{F}_i - 2\vec{\Omega}\times\vec{U} + \Omega^2\vec{R} \qquad (2.2)$$

where $\vec{U}$ is the velocity, $\sum\vec{F}_i$ is the sum of the real forces per unit mass and $\vec{R}$ is the position vector from the axis of rotation (perpendicular to the axis) to the point in question.

(c) Resolve the 3D Coriolis force $-2\vec{\Omega}\times\vec{U}$ into its components in the directions of the unit vectors $(\vec{i},\vec{j},\vec{k})$ (notation conventional). Hence, show that the sum of the 3D Coriolis force and the horizontal pressure gradient force is exactly zero if $\vec{U}$ lies in the meridional direction, $\vec{\nabla}_H p$ lies in the zonal direction and the meridional wind component v equals its geostrophic value, v_g. [Note that no scaling of eq. (2.2) is necessary to achieve this result.]

Question 3.

(a) Assuming horizontal motion on a tangent plane to the Earth, show that in natural coordinates the acceleration $D\vec{V}/Dt$ can be written

$$\frac{D\vec{V}}{Dt} = \frac{DV}{Dt}\vec{t} + \frac{V^2}{R}\vec{n} \qquad (3.1)$$

where $\vec{V}$ is the horizontal velocity vector, $(\vec{t}, \vec{n})$ are unit vectors with $\vec{t}$ in the direction of the flow and $\vec{n}$ normal to $\vec{t}$ and oriented to the left of the flow direction, and R is the radius of curvature (positive when the centre of curvature is in the positive $\vec{n}$ direction).

(b) Write down the horizontal momentum equation in vector form for frictionless horizontal flow on an f-plane, using the isobaric vertical coordinate system. Using (3.1), express this equation in its components in natural coordinates.
Hint: the horizontal pressure gradient force in the isobaric coordinate system is $-\nabla_p \Phi$, where Φ is the geopotential.

(c) What is cyclostrophic flow? Using the $\vec{n}$-component of the momentum equation in natural coordinates derived above, find an expression for the cyclostrophic wind speed. Is the flow direction cyclonic or anticyclonic? Supposing a tornado has a tangential velocity of 40 ms^{-1} at a distance of 250m from the centre of the vortex, show that the condition for cyclostrophic flow is satisfied. (Confine attention to the Northern Hemisphere and take $f = 10^{-4}\ s^{-1}$.)

Question 4.

a) State the assumptions used in constructing the shallow water model with a free surface of height $h_T(x,y,t)$, bottom topography $h_s(x,y)$ and fluid depth $h(x,y,t)$ on a β-plane.

b) Starting from the primitive equations of motion on a β-plane and using the above assumptions, show that the governing equations for the model in question are

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \qquad (4.1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \qquad (4.2)$$

$$\frac{Dh}{Dt} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (4.3)$$

where $\Phi_T = gh_T$.

c) Derive the linearized form of the above equations for small perturbations about a state of rest for the case where $h_s = 0$ and $\beta = 0$. Hence show the existence of 1D gravity-inertia wave solutions with phase speed

$$c = \pm\left(gH + \frac{f_0^2}{k^2}\right)^{1/2} \qquad (4.4)$$

where H is the mean depth of the fluid and k is the wavenumber in x.

If H=1km, for what value of the wavelength are the gravity and inertia terms in (4.4) equal? (Take $g = 9.81\text{ms}^{-2}$ and $f_0 = 10^{-4}\,\text{s}^{-1}$.)

Question 5.

a) The potential vorticity equation for the shallow water model with a free surface and a flat bottom on an f-plane can be written in conventional notation as

$$\frac{D}{Dt}\left[\frac{\varsigma + f_0}{\Phi}\right] = 0 \qquad (5.1)$$

Show that the linearized version of (5.1) for small perturbations about a state of rest can be written

$$\frac{\partial}{\partial t}\left(\varsigma' - \frac{f_0}{\overline{\Phi}}\Phi'\right) = 0 \qquad (5.2)$$

b) At an initial time t=0 the free surface height of the above shallow water model is given by

$$h = H + h_0, \text{x} < 0 \qquad (5.3)$$
$$h = H - h_0\,, \text{x} > 0 \qquad (5.4)$$

and the initial velocity is zero everywhere. Using (5.2) show that the equations determining the free surface height perturbations at t=∞, when the system has adjusted to geostrophic balance, are

$$\frac{d^2}{dx^2}(h' - h_0) = \frac{1}{R^2}(h' - h_0),\ x < 0 \qquad (5.5)$$
$$\frac{\mathrm{d}^2}{\mathrm{dx}^2}(h' + h_0) = \frac{1}{R^2}(h' + h_0),\ \mathrm{x} > 0 \qquad (5.6)$$

where $R = \sqrt{\overline{\Phi}}\,/\,f_0$.

(c) Solve (5.5) and (5.6) subject to the appropriate boundary conditions to obtain the free surface height at t=∞.

Question 6.

(a) The quasi-geostrophic potential vorticity equation for frictionless adiabatic flow in z^* [= - H ln(p/p$_s$)] coordinates on a β-plane (f= f_0 +βy) can be written

$$\left(\frac{\partial}{\partial t} + \vec{V}_g \bullet \vec{\nabla}\right)q = 0 \qquad (6.1)$$

where

$$q = \nabla^2\psi + f + \frac{1}{\rho_0}\frac{\partial}{\partial z^*}\left(\frac{f_0^{\,2}}{N^2}\rho_0\frac{\partial\psi}{\partial z^*}\right) \qquad (6.2)$$

In the above, ψ is the geostrophic streamfunction, ∇^2 is the horizontal Laplacian, $\vec{V}_g = \vec{k}\times\vec{\nabla}\psi$, N is the buoyancy frequency and ρ_0 is the reference density, given by

$$\rho_0 = \rho_s \exp(-z^* / H) \qquad (6.3)$$

with ρ_s and H both constants.

Show that the linearized form of (6.1) for a small perturbation about a constant mean wind $\bar{u}$ can be written

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + \beta\frac{\partial \psi'}{\partial x} = 0 \qquad (6.4)$$

where

$$q' = \nabla^2\psi' + \frac{1}{\rho_0}\frac{\partial}{\partial z^*}\left(\frac{f_0^2}{N^2}\rho_0\frac{\partial \psi'}{\partial z^*}\right) \qquad (6.5)$$

Hint: assume $\psi = -\bar{u}y + \psi'(x, y, z^*, t)$.

(b) Consider the case of a perturbation forced by the mean wind blowing over topography of the form

$$h = h_s \cos ly\, e^{ikx} \qquad (6.6)$$

Assuming N is constant, show that a steady-state solution to (6.4) that is oscillatory solution in z^* (and can therefore propagate wave energy vertically) is possible only if the mean wind $\bar{u}$ lies between zero and an upper limit U_c given by

$$U_c = \frac{\beta}{\left(k^2 + l^2\right) + \frac{1}{4L_R^2}} \qquad (6.7)$$

where L_R ($=NH/f_0$) is the Rossby radius of deformation.
Hint: Assume the solution has the form

$$\psi' = \Psi(z^*)\exp(z^* / 2H)\cos ly\, \mathrm{e}^{ikx} \qquad (6.8)$$

Note: you are not required to make explicit use of (6.6) to find the full solution satisfying the lower boundary condition here. The form of the topography is given only to indicate why the topographically forced solution should have the x and y-dependence given in (6.8).

(c) Find the value of U_c if the β-plane is tangent to the Earth at 45° latitude, L_R = 1000 km, $l = 0$ and the x-wavelength is that corresponding to zonal wavenumber 1 at the latitude of tangency. ($\Omega = 7.29 \times 10^{-5}\ \mathrm{s}^{-1}$, a = 6370 km.)

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