University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER I EXAMINATION 2007/2008

MAPH 40230

Dynamic Meteorology

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Head of School: Prof Séan Dineen

Examiner: Prof Ray Bates*

Time Allowed: 3 hours

Instructions for Candidates

Full marks will be awarded for complete answers
to **four** questions.
All questions carry equal marks.
Please do not use red pen on the answer books.
Please use separate answer book for each question.

Instructions for Invigilators

Non-programmable calculators may be used during this examination.

Question 1.

(a) The horizontal momentum equation for frictionless flow on a tangent plane to the Earth can be written in isobaric coordinates as

$$\frac{D\vec{V}}{Dt} = -f\vec{k} \times \vec{V} - \nabla\Phi \qquad (1.1)$$

where the notation is conventional.

For horizontal motion, $D\vec{V}/Dt$ can be written in natural coordinates as

$$\frac{D\vec{V}}{Dt} = \frac{DV}{Dt}\vec{t} + \frac{V^2}{R}\vec{n} \qquad (1.2)$$

where $(\vec{t}, \vec{n})$ are unit vectors with $\vec{t}$ in the direction of the flow and $\vec{n}$ normal to $\vec{t}$ and oriented to the left of the flow direction, and R is the radius of curvature (positive when the centre of curvature is in the positive $\vec{n}$ direction). Using (1.2), show that the components of (1.1) in the $\vec{t}$ and $\vec{n}$ directions are

$$\frac{DV}{Dt} = -\frac{\partial\Phi}{\partial s} \qquad (1.3)$$

$$\frac{\mathrm{V}^2}{\mathrm{R}} = -fV - \frac{\partial\Phi}{\partial n} \qquad (1.4)$$

(b) Using (1.3) and (1.4), obtain an expression for the gradient wind speed V (assume steady circular flow parallel to the height contours on an f-plane).

(c) Using this expression, show that the geopotential gradient in a regular high must approach zero as $|R| \rightarrow 0$, but that no such restriction exists for a regular low. Hint: Use the condition that V be real. Confine attention to the Northern Hemisphere.

Question 2.

a) What physical principle is embodied in the continuity equation? Using the Lagrangian control volume method, derive the continuity equation in (x,y,z,t) coordinates in the form

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{U} = 0 \qquad (2.1)$$

where ρ is the density and U is the 3D velocity vector.

b) Again using the Lagrangian control volume method, derive the continuity equation in isobaric coordinates for a hydrostatic atmosphere. Comment on the advantages of this equation.

Question 3.

a) State the assumptions used in constructing the shallow water model with a rigid horizontal upper lid, bottom topography $h_s(x,y)$ and fluid depth $h(x,y)$ on a β-plane.

b) Starting from the primitive equations of motion on a β-plane and using the above assumptions, show that the governing equations for the model in question are

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \qquad (3.1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \qquad (3.2)$$

$$\frac{Dh}{Dt} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3.3)$$

where $\Phi_T = p_T/\rho$, p_T being the pressure at the upper lid and ρ being the density.

c) Using (3.1) and (3.2) derive the vorticity equation

$$\frac{D}{Dt}(\zeta + \mathrm{f}) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3.4)$$

for this model (ζ = relative vorticity).

Note: you may make use of the relationship

$$\frac{\partial}{\partial x}\left(\vec{V} \bullet \nabla v\right) - \frac{\partial}{\partial y}\left(\vec{V} \bullet \nabla u\right) = \vec{V} \bullet \nabla \zeta + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

where $\vec{V}$ is the horizontal velocity.

Question 4.

a) For the shallow water model with a rigid lid on a β-plane, show that when the bottom topography is flat the vorticity equation (3.4) can be written in terms of the streamfunction ψ as

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi\right) + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0 \qquad (4.1)$$

where ∇^2 is the horizontal Laplacian and J is the Jacobian.

Hint: make use of equation (3.3).

b) Using (4.1) show that the frequency of 2D Rossby waves superimposed on a basic current $\bar{u}$ in this model is

$$\nu = \bar{u}k - \frac{\beta k}{K^2} \qquad (4.2)$$

where $K^2 = k^2+l^2$ and (k,l) are the wavenumbers in (x,y).
Hint: assume $\psi = \overline{\psi}(y) + \psi'(x, y, t)$ and linearize.
If the x-wavelength is 10,000 km, the y-wavelength is infinite and $\beta = 1.6 \times 10^{-11} m^{-1} s^{-1}$, what value of $\overline{u}$ is required to reduce the phase speed of the Rossby wave to zero?

c) Calculate the group velocity for the waves described by (4.2).

Question 5.

(a) Show that in a hydrostatic atmosphere the internal and gravitational potential energies are proportional and that the sum of these two forms of energy (i.e., the total potential energy – TPE) can be written

$$TPE = \left(\frac{c_p}{c_v}\right) E_I \qquad (5.1)$$

where c_p and c_v are the specific heats of air at constant pressure and constant volume, respectively, and E_I is the internal energy.

(b) Show that the TPE of a unit column of atmosphere of uniform potential temperature θ extending from the surface ($p=p_0$) to the top of the atmosphere ($p=0$) is

$$TPE = \frac{c_p}{g} \frac{p_0 \theta}{\kappa + 1} \qquad (5.2)$$

where $\kappa = R / c_p$ and g is the acceleration of gravity.

(c) Consider two air masses of uniform potential temperatures θ_1 and θ_2 ($\theta_2 > \theta_1$) which are separated by a vertical partition. Each air mass occupies a horizontal area A and extends from the surface to the top of the atmosphere. Show that the available potential energy for this system is given by

$$APE = \frac{c_p}{g} \frac{p_0}{\kappa + 1} \left(1 - \frac{1}{2^\kappa}\right) (\theta_2 - \theta_1) A \qquad (5.3)$$

Question 6.

(a) Given the following governing equations for the two-layer model of baroclinic instability

$$\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_1'}{\partial x^2}+\beta\frac{\partial\psi_1'}{\partial x}=\frac{f_0}{\delta p}\omega_2' \qquad (6.1)$$

$$\left(\frac{\partial}{\partial t}+U_3\frac{\partial}{\partial x}\right)\frac{\partial^2\psi_3'}{\partial x^2}+\beta\frac{\partial\psi_3'}{\partial x}=-\frac{f_0}{\delta p}\omega_2' \qquad (6.2)$$

$$\left(\frac{\partial}{\partial t}+U_m\frac{\partial}{\partial x}\right)\left(\psi_1'-\psi_3'\right)-U_T\frac{\partial}{\partial x}\left(\psi_1'+\psi_3'\right)=\frac{\sigma\delta p}{f_0}\omega_2' \qquad (6.3)$$

where $U_m = (U_1+U_3)/2$ and $U_T=(U_1-U_3)/2$, show that the rate of change of the sum of the kinetic and available potential energies is

$$\frac{d}{dt}(K'+P')=4\lambda^2 U_T\overline{\psi_T\frac{\partial\psi_m}{\partial x}} \qquad (6.4)$$

where $\psi_m = \left(\psi_1'+\psi_3'\right)/2$, $\psi_T=\left(\psi_1'-\psi_3'\right)/2$, $\lambda^2=(f_0)^2/\left[\sigma(\delta p)^2\right]$ and $\overline{(\ )}$ denotes an average over the wavelength of the disturbance.

(b) Discuss the energetics of baroclinic waves qualitatively using a box diagram involving $\overline{P}$ (the mean available potential energy), P' and K'.

(c) Discuss the physical mechanism of baroclinic instability in terms of the slope of particle trajectories ('the wedge of instability').

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