University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SEMESTER I EXAMINATION 2006/2007

MAPH 40230

Dynamical Meteorology

Extern examiner: Prof. Frank Hodnett
Head of School: Prof. Adrian Ottewill
Examiner: Prof. Ray Bates*

Time Allowed: 3 hours

Instructions for Candidates

Answer **four (4)** of the following six questions. Each question carries 25 marks.

Instructions for Invigilators

Non-programmable calculators may be used during this examination.

Question 1.

(a) Assuming horizontal motion on a tangent plane to the Earth, show that in natural coordinates the acceleration $D\vec{V}/Dt$ can be written

$$\frac{D\vec{V}}{Dt} = \frac{DV}{Dt}\vec{t} + \frac{V^2}{R}\vec{n} \qquad (1.1)$$

where $\vec{V}$ is the horizontal velocity vector, $(\vec{t}, \vec{n})$ are unit vectors with $\vec{t}$ in the direction of the flow and $\vec{n}$ normal to $\vec{t}$ and oriented to the left of the flow direction, and R is the radius of curvature (positive when the centre of curvature is in the positive $\vec{n}$ direction).

(b) Using the results of (a), write down the horizontal momentum equation for horizontal flow in natural coordinates on an f-plane, assuming an isobaric vertical coordinate.
Hint: the horizontal pressure gradient force in the isobaric coordinate system is $-\nabla\Phi$, where Φ is the geopotential.

(c) Show that if the horizontal pressure gradient force is zero, steady inertial motion in a circle is possible. What is the direction of the flow and what is its period?

Question 2.

(a) Define potential temperature (θ). Show that when θ is constant with height (z) the lapse rate of temperature is given by

$$\Gamma_d \equiv -\frac{dT}{dz} = \frac{g}{c_p} \qquad (2.1)$$

where g is the acceleration of gravity and c_p is the specific heat of dry air at constant pressure. (Assume the atmosphere is hydrostatically balanced.)

(b) Given the thermodynamic energy equation in the form

$$c_v \frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J \qquad (2.2)$$

where c_v is the specific heat of dry air at constant volume, α is the specific volume and J the diabatic heating rate per unit mass, show that this equation can be written in terms of potential temperature as

$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T} J \qquad (2.3)$$

(Use the standard thermodynamic relationship $R = c_p - c_v$).
Hint: You may work backwards from (2.3) using the definition of θ if you wish.

(c) Show that in the isobaric coordinate system eq. (2.2) can be written in the form

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p} \qquad (2.4)$$

where

$$S_p = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} \qquad (2.5)$$

Note: $\omega \equiv Dp/Dt$.

Question 3.

a) State the assumptions used in constructing the shallow water model with a free surface of height $h_T(x,y,t)$, bottom topography $h_s(x,y)$ and fluid depth $h(x,y,t)$ on a β-plane.

b) Starting from the primitive equations of motion on a β-plane and using the above assumptions, show that the governing equations for the model in question are

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \qquad (3.1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \qquad (3.2)$$

$$\frac{Dh}{Dt} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3.3)$$

where $\Phi_T = gh_T$.

c) Derive the linearized form of the above equations for small perturbations about a state of rest for the case where $h_s = 0$ and $\beta = 0$. Hence show the existence of 1D gravity-inertia wave solutions with phase speed

$$c = \pm\left(gH + \frac{f_0^2}{k^2}\right)^{1/2} \qquad (3.4)$$

where H is the mean depth of the fluid and k is the wavenumber in x.

Question 4.

(a) The barotropic vorticity equation (or vorticity equation for the shallow water model with a rigid lid and flat bottom) on a β-plane is given in terms of the streamfunction ψ by

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + J(\psi, \nabla^2\psi) + \beta\frac{\partial \psi}{\partial x} = 0 \qquad (4.1)$$

Linearize the above equation about a state of rest and hence show that the frequency of unbounded 2D Rossby waves is given by

$$\nu = -\frac{\beta k}{K^2} \qquad (4.2)$$

where k is the wavenumber in x and K is the magnitude of the 2D wavenumber vector.

(b) Show that the above unbounded 2D Rossby wave is a transverse wave.

(c) Suppose we have a channel on a β-plane bounded by frictionless vertical walls at y = 0 and y = D. Again using the linearized form of (4.1) for small perturbations about a state of rest, show that a wave solution of the form $\psi' = \hat{\psi}(y)\cos k(x-ct)$ exists within the channel for which the phase speed is given by

$$c = -\frac{\beta}{k^2 + \left(\dfrac{n\pi}{D}\right)^2}, \quad n = 1,2,3,\ldots\ldots \qquad (4.3)$$

Hint: Assume the y-dependence of $\hat{\psi}(y)$ is sinusoidal, not exponential.

Question 5.

(a) Write down the governing momentum and continuity equations for the shallow water model with a rigid horizontal lid, mean depth H, bottom topography of height $h_s(x,y)$ and fluid depth h(x,y) on a β-plane.

(b) Derive the quasi-geostrophic form of these equations.
Note: use constant-f geostrophy in defining the geostrophic current.

(c) Hence show that the quasi-geostrophic potential vorticity equation for this model can be written

$$\frac{D_g}{Dt}\left[\zeta_g + f + \frac{f_0}{H}h_s\right] = 0 \qquad (5.1)$$

Question 6.

(a) In the Eady model of baroclinic instability, the solution for the perturbation streamfunction is given by

$$\psi' = A\left[\sinh \alpha z^* - \frac{c\alpha}{\Lambda}\cosh \alpha z^*\right]\cos ly \exp[ik(x-ct)] \qquad (6.1)$$

where the complex phase speed c (= $c_r + ic_i$) is given by

$$c = \frac{\Lambda H}{2}\left[1 \pm \frac{2}{\alpha H}\left\{\left(\frac{\alpha H}{2} - \tanh\frac{\alpha H}{2}\right)\left(\frac{\alpha H}{2} - \coth\frac{\alpha H}{2}\right)\right\}^{1/2}\right] \qquad (6.2)$$

In the above, Λ is the vertical wind shear, H the height of the lid and $\alpha = N(k^2 + l^2)^{1/2}/f_0$, N being the buoyancy frequency and f_0 the (constant) Coriolis parameter.

For the case $l = 0$, show that in the limit $L_x / L_R \to \infty$ (where $L_x = 2\pi/k$ and $L_R = NH/f_0$), the complex phase speed for the unstable wave is given by

$$c = \frac{\Lambda H}{2}\left(1 + \frac{i}{\sqrt{3}}\right) \qquad (6.3)$$

[Make use of the series expansions

tanh x = x – x^3/3 + ...
coth x = 1/x + x/3 - .....]

(b) For the same case $l = 0$ and in the limit $L_x / L_R \to \infty$, show that the solution for ψ' given by (6.1) reduces in the case of the unstable wave to

$$\psi' = AkL_R\left[\frac{z^*}{H} - \frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right)\right]\exp[ik(x - ct)] \qquad (6.4)$$

[In deriving (6.4), make use of the series expansions
sinh x = x + x^3/3! + ...
cosh x = 1+ x^2/2! + ...]

(c) Does the perturbation streamfunction given by (6.4) slope eastward or westward with height? Qualitatively, what does the direction of vertical slope of the streamfunction tell us about the direction of the eddy heat transport?

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