Mathematics & Music: Symmetry & Symbiosis

Peter Lynch
School of Mathematics & Statistics
University College Dublin

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Outline

- “The Two Cultures”
- Pythagoras
- Sinusoidal Waves
- Musical Notation
- Tuning
- Canons & Fugues
- Fascinating Rhythm
- Symmetry
- Musical Chords

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Three Sweeping Statements: No. 1

Everyone Loves Music

Three Sweeping Statements: No. 2

Everyone Hates Maths

Three Sweeping Statements: No. 3

Music and Mathematics are the Same Thing

Love/Hate Image

Given the close link between music and maths, how can we love one and hate the other?

1959 Rede Lecture: *The Two Cultures*

The concept of The Two Cultures was introduced by the British scientist and novelist C. P. Snow.

This concept is still relevant today.

Who's Who Here?

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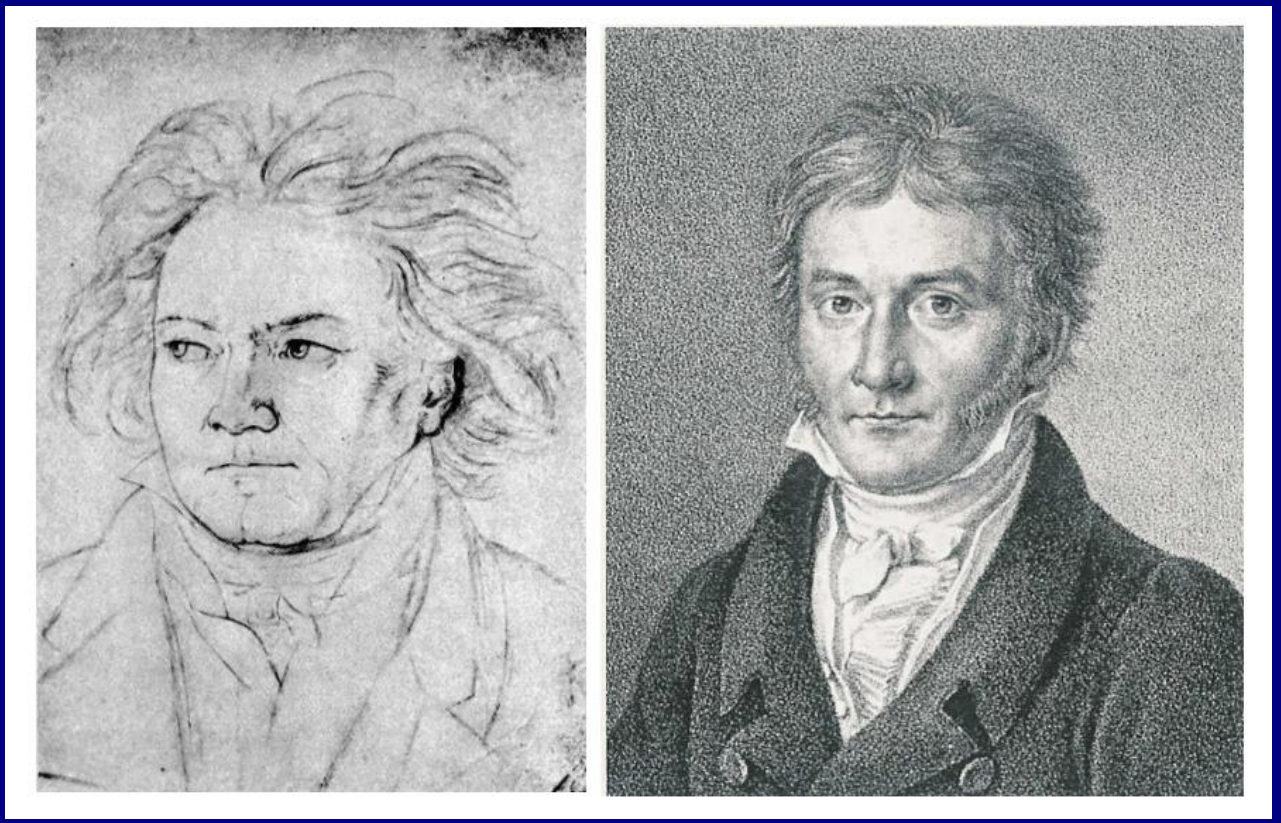
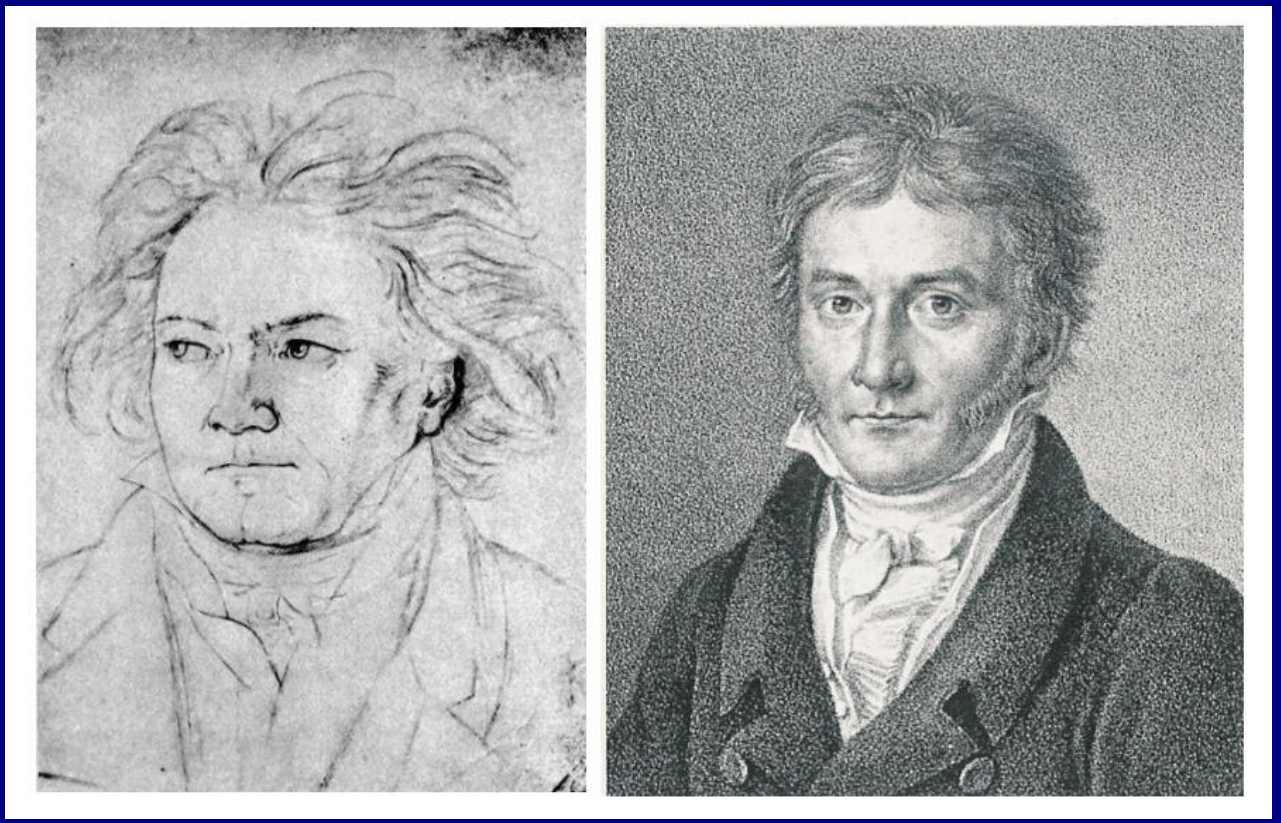


LEFT: Ludwig van Beethoven (1770–1827)

Who's Who Here?



LEFT: Ludwig van Beethoven (1770–1827)
RIGHT: Carl Friedrich Gauss (1777–1855)

Beethoven / Gauss

Beethoven and Gauss were at the height of their creativity in the early 19th century.

The work of Gauss has a greater impact on our daily lives than the magnificent creations of Beethoven.

Yet, Beethoven is known to all, Gauss to only a few!

"Of course I've heard of Beethoven, but who is this Gauss dude?"

Two Parallel Languages

MUSIC	MATHS
Pitch	Frequency
Scales	Modular Arithmetic
Intervals	Logarithms
Overtones	Integers
Octave Identification	Equivalence Relation
Equal Temperament	Exponents
Timbre	Harmonic Analysis
Canon Form	Group Theory
Chord Progressions	Orbifold Topology

You know more mathematics than you realize!

Music & Maths

There are many parallels between music and maths:

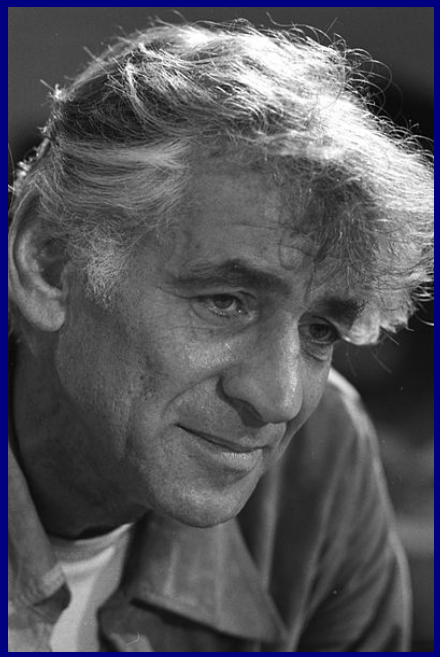
Structure Symmetry Pattern etc.

But music is accessible to all while maths is not.
Music gets into the soul through the emotions.

Maths is understood through the intellect.
Appreciation comes via a rational route.

Music has instant appeal. Maths takes time.

Leonard Bernstein



"Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?"

Some Relevant Mathematical Concepts

- Integers. Rationals. Real Numbers
- Logarithms and Exponentials
- Equivalence Relationships
- Geometric Transformations
- Modular Arithmetic
- Groups and Rings
- Periodic Functions
- Orbifold Topology.

Outline

The Quadrivium

The Pythagoreans organized their studies into the Quadrivium, comprising four disciplines:

- **Arithmetic**
- **Geometry**
- **Music**
- **Astronomy**

Static/Dynamic Number. Pure/Applied

- **Arithmetic:** Static number
- **Music:** Dynamic number

Arithmetic represents numbers at rest.

Music is numbers in motion.

Arithmetic is **pure or abstract** in nature.

Music is **applied or concrete** in nature.

Static/Dynamic Space. Pure/Applied

- Geometry: Static space
- Astronomy: Dynamic space

Geometry represents space at rest.

Astronomy is space in motion.

Geometry is pure or abstract in nature.

Astronomy is applied or concrete in nature.

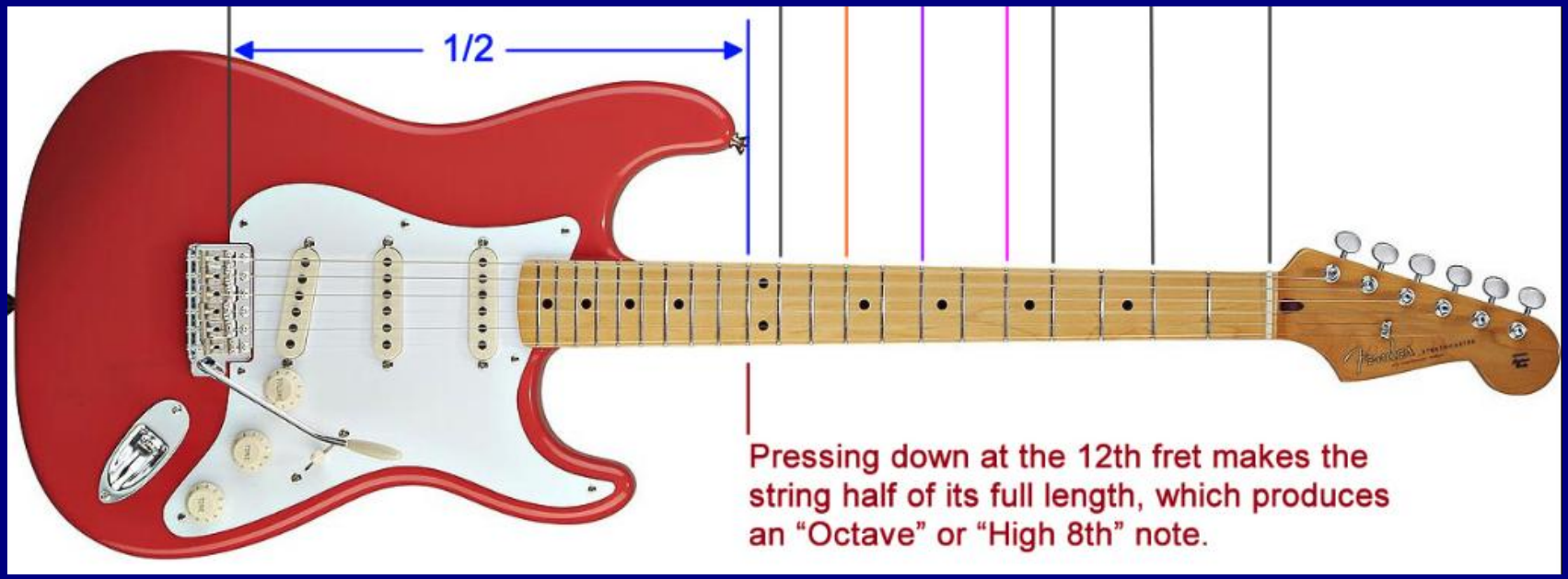
Discovery of Pythagoras

The Pythagoreans discovered a remarkable connection:

Ratios of small whole numbers are directly linked with consonant or harmonically pleasing chords.

"There is geometry in the humming of the strings, There is music in the spacing of the spheres."

Guitar Strings

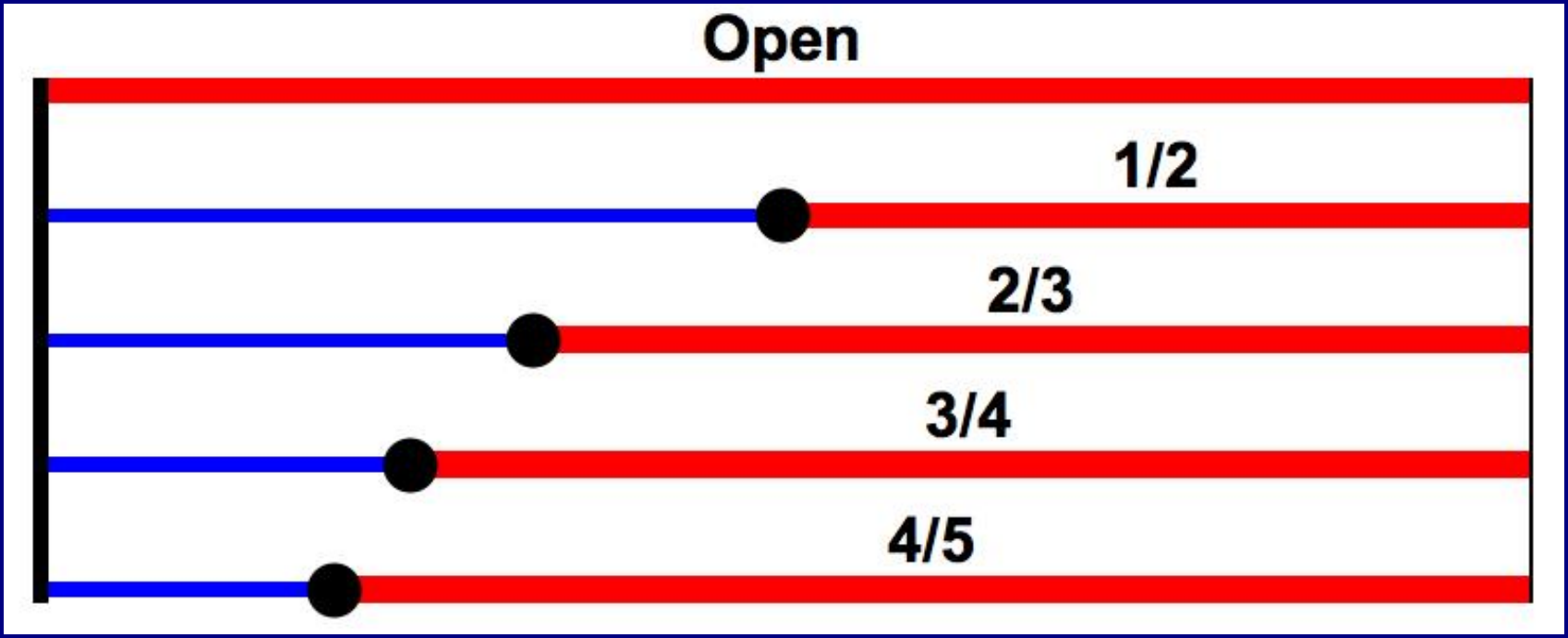


Open string vibrates at 264 Hz. Call this f.

- **String of length $\frac{1}{2}$ vibrates at $\frac{2}{1}f$.**
- **String of length $\frac{2}{3}$ vibrates at $\frac{3}{2}f$.**
- **String of length $\frac{3}{4}$ vibrates at $\frac{4}{3}f$.**

Demonstrate the pitches on the uke.

The Monochord



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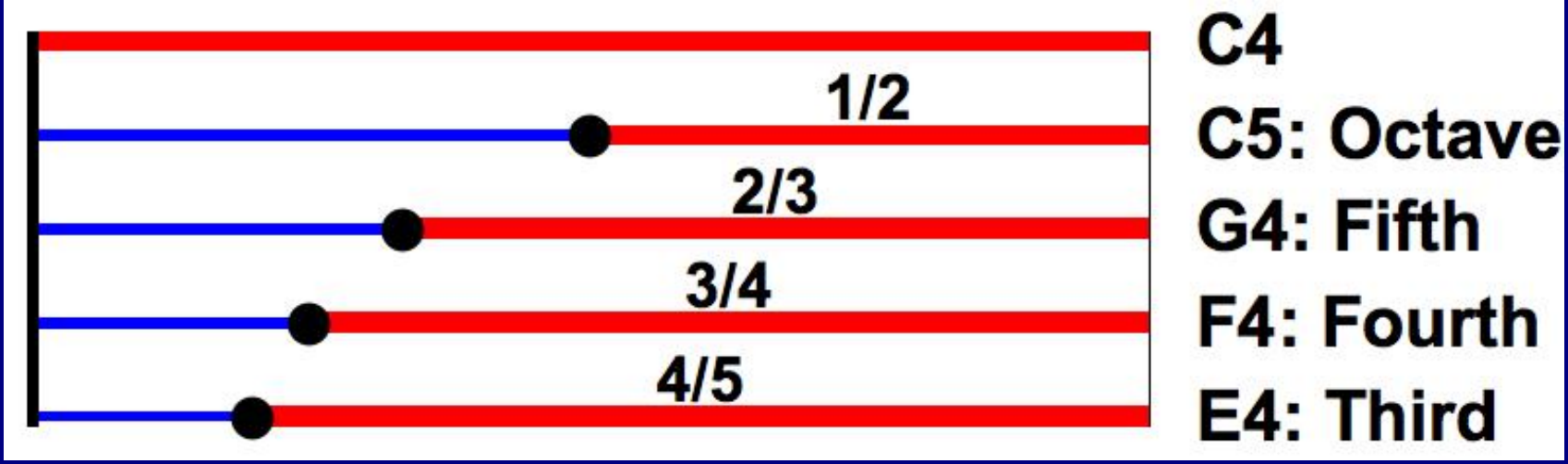


Feynman on Pythagoras' Discovery

Richard Feynman

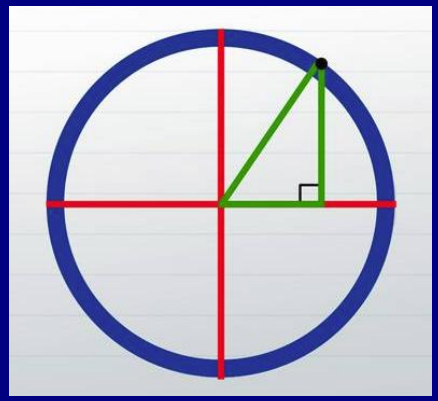
"The first example, outside geometry, of a numerical relationship in nature."

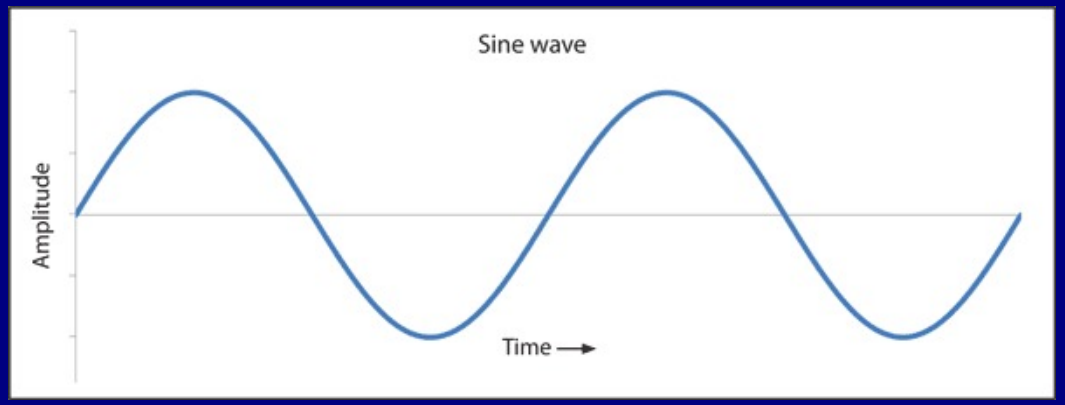
Pythagoras made his discovery through observation.
This aspect does not seem to have impressed him.

Had Pythagoras followed up on this idea,
"Physics might have had a much earlier start."

Outline

Sine Waves and Circles





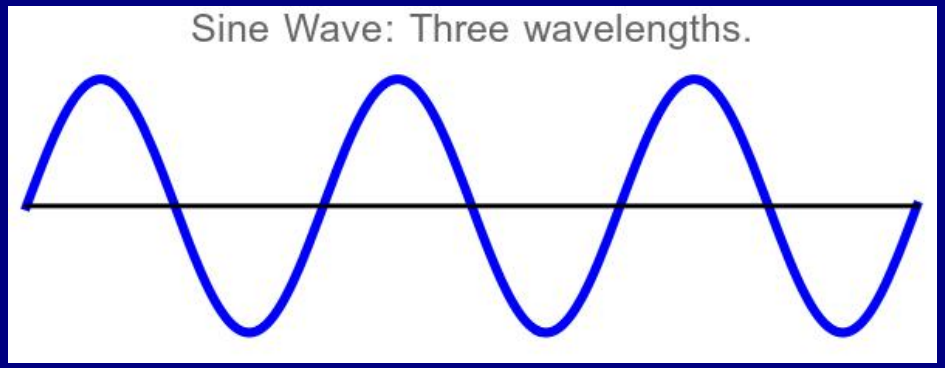
As the black dot moves around the circle, its height traces a sine wave in time.

Up and down and up and down and up and down

and up and down and up and down and . . .

Sine waves are a kind of circular functions.

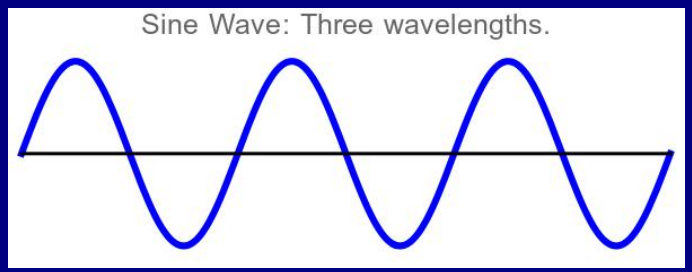
Teerminology

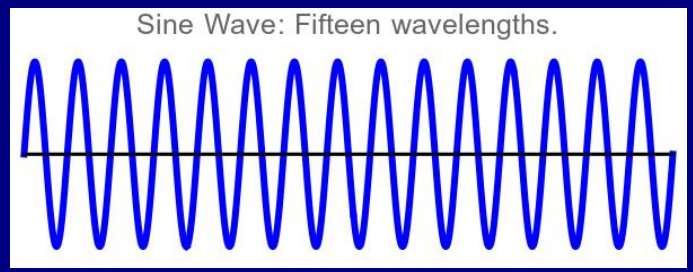


Music **Physics**

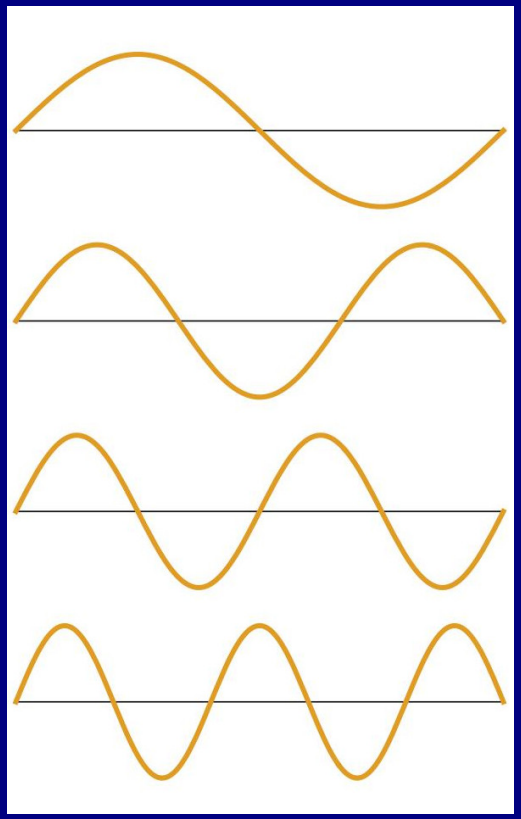
Loudness <=> Amplitude

Pitch <=> Frequency



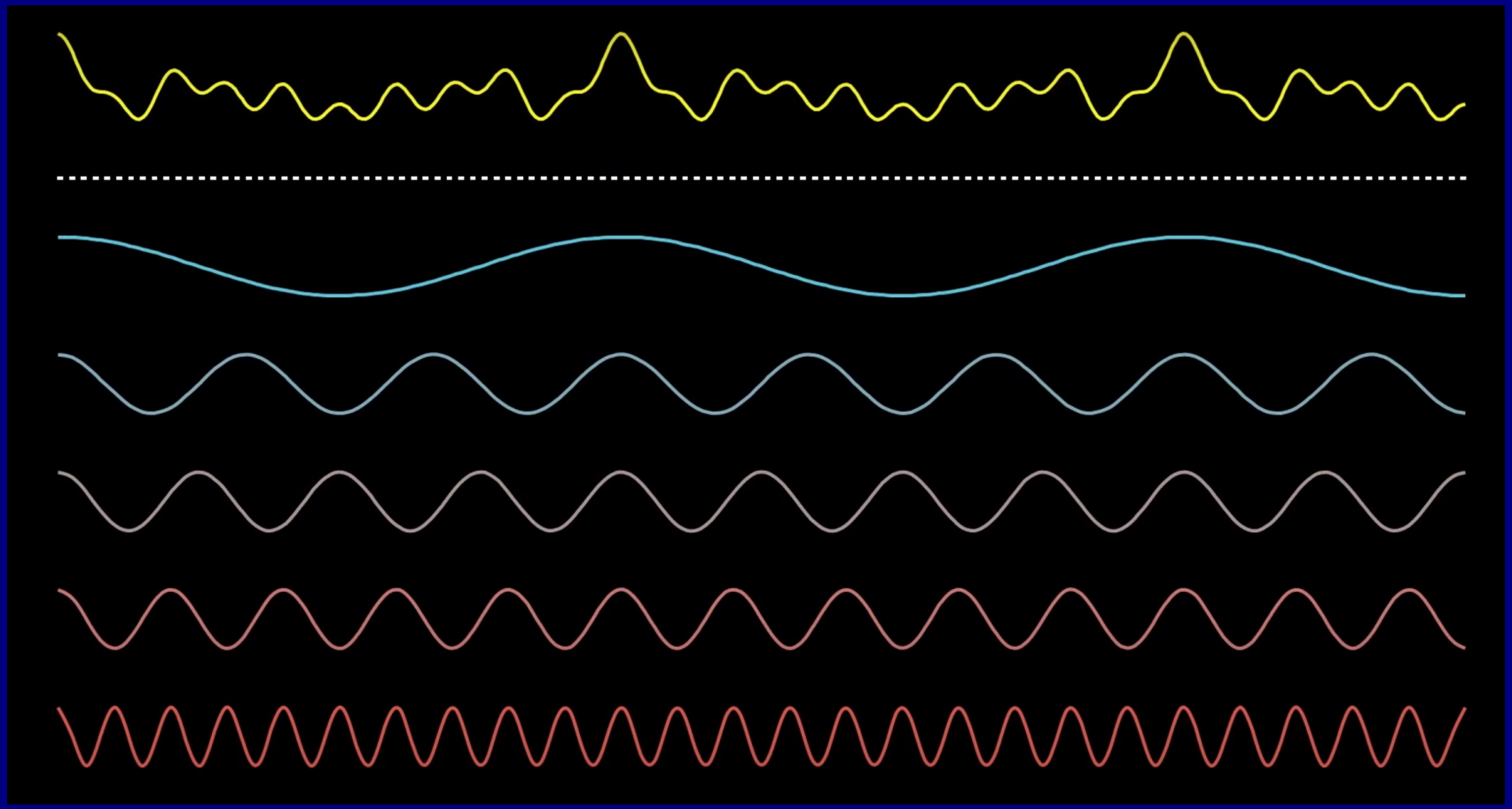


Simple Harmonics



A simple sine wave and its first three overtones.

Fourier Components



A Harmonic Generator

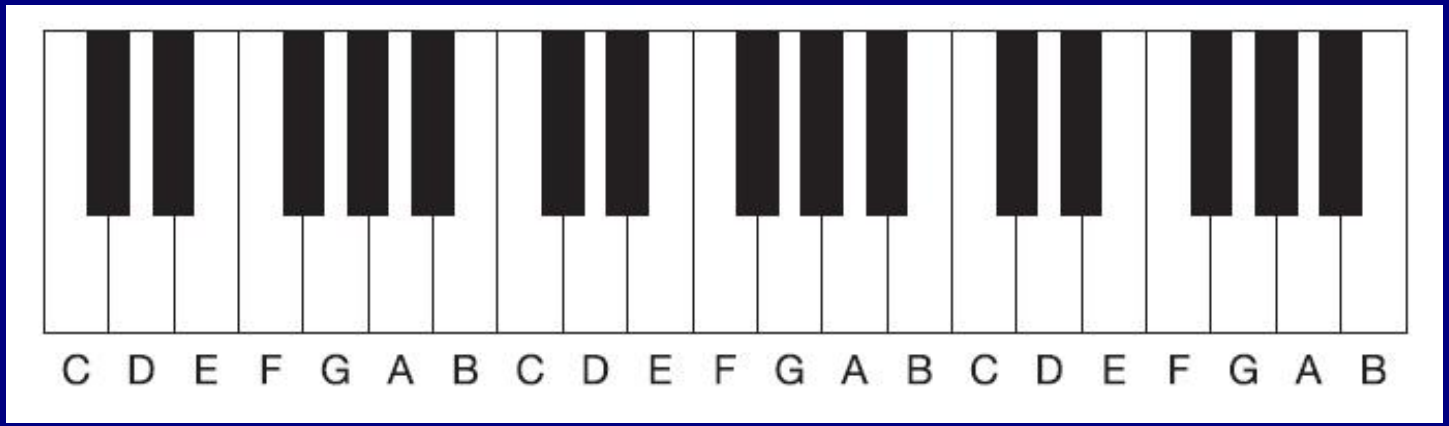
https://meettechniek.info/additional/additive-synthesis.html

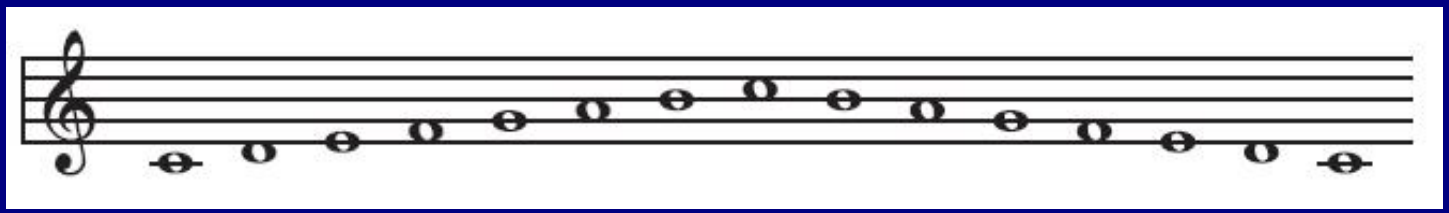
A website for generating sine waves and harmonics.

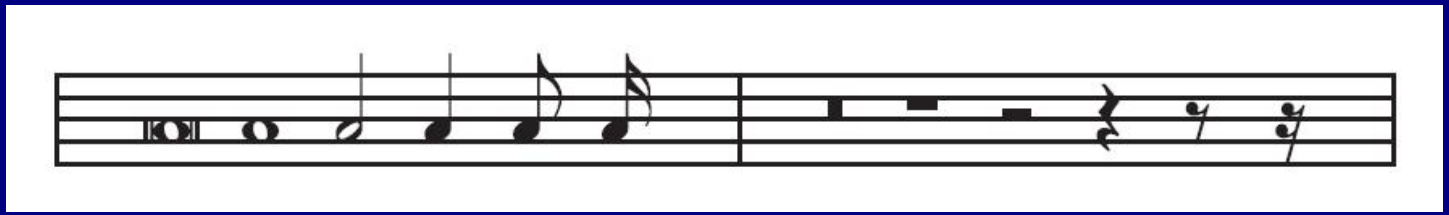
*** * * If the technology fails, use the trusty Uke * * ***

Outline

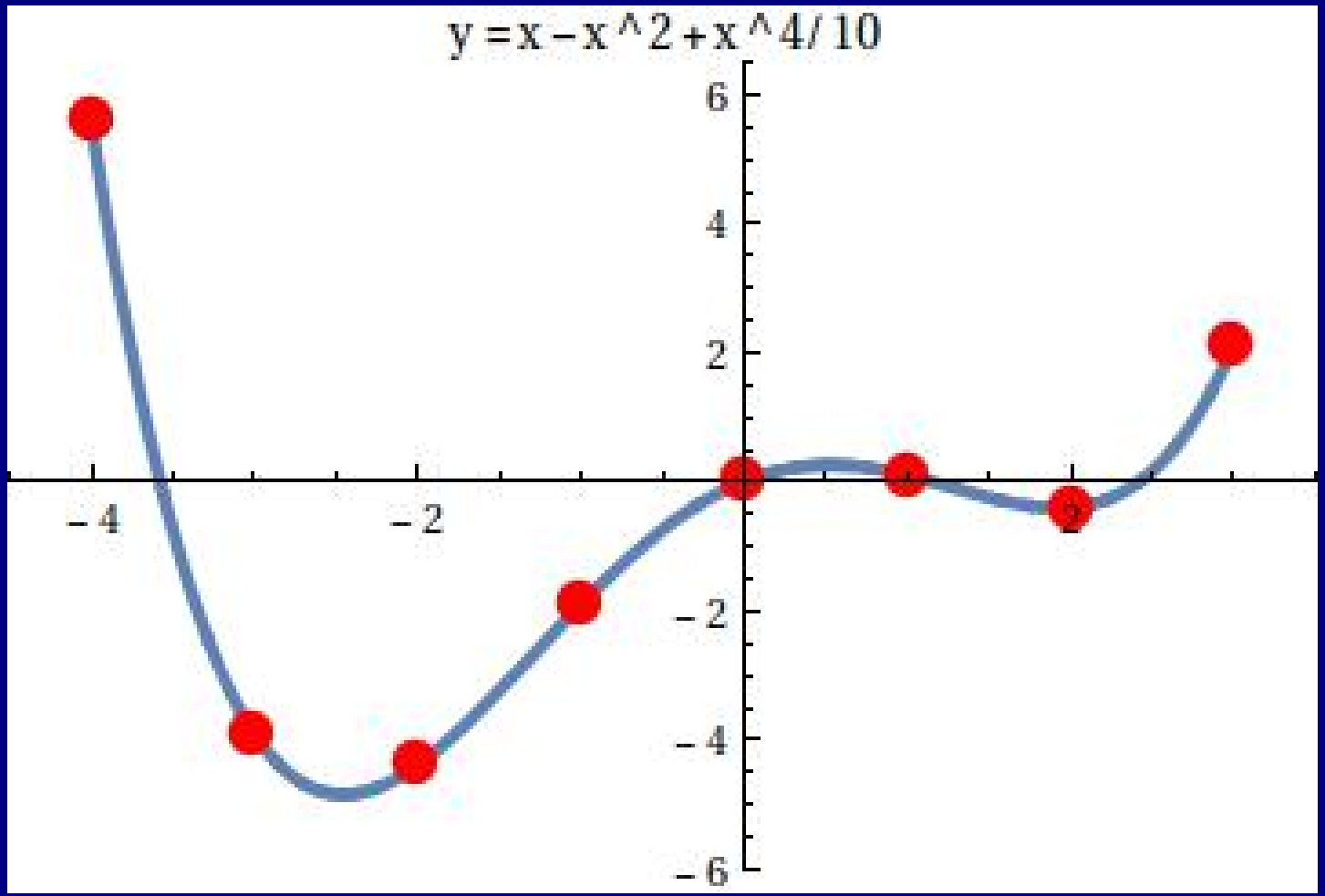
Notating Musical Notes





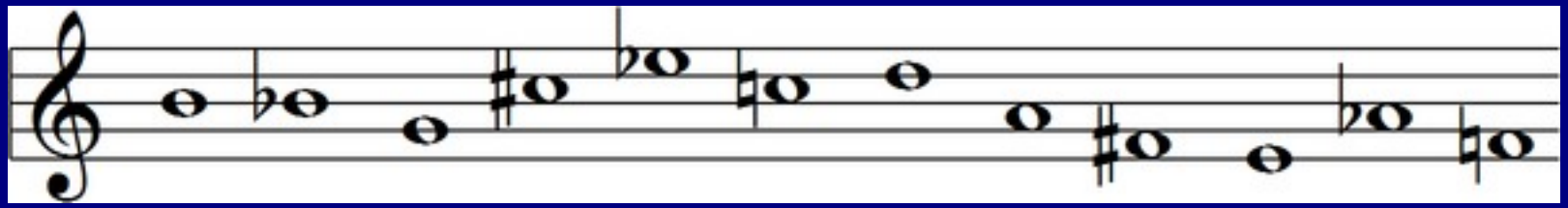


A Mathematical Graph: Joining the Dots

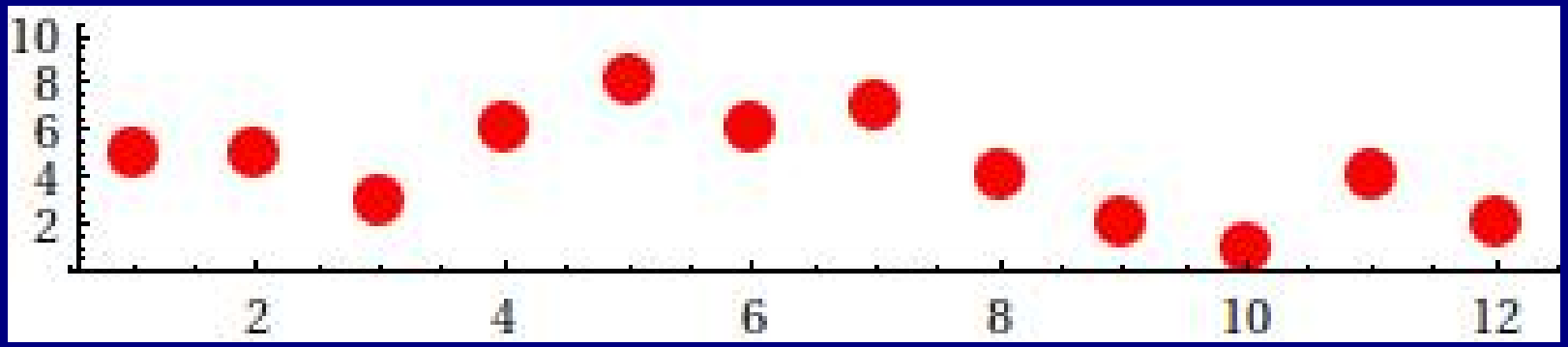


A plot of y versus x.

Music as a Graph

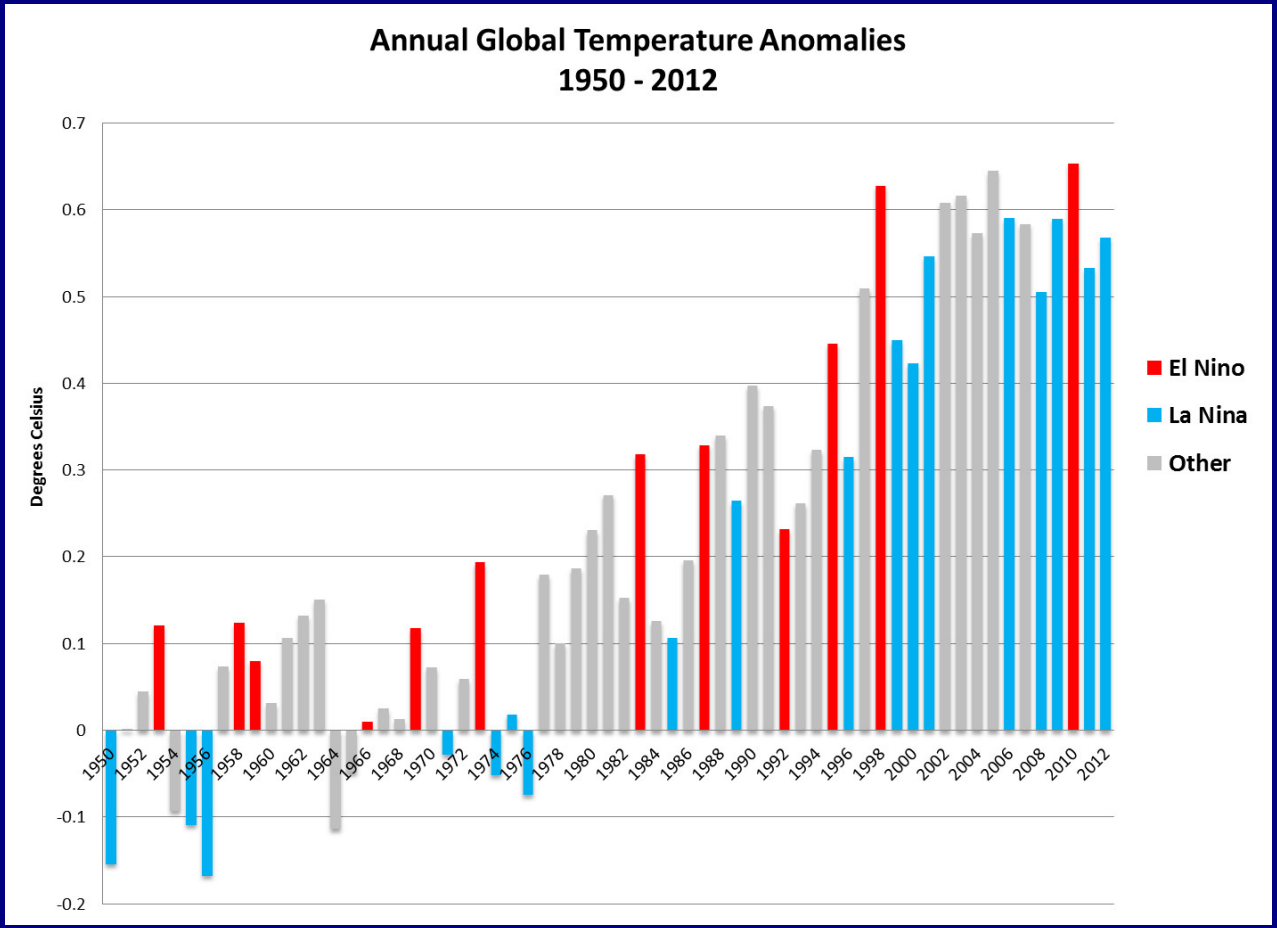


A sequence of musical notes: a simple tone row.



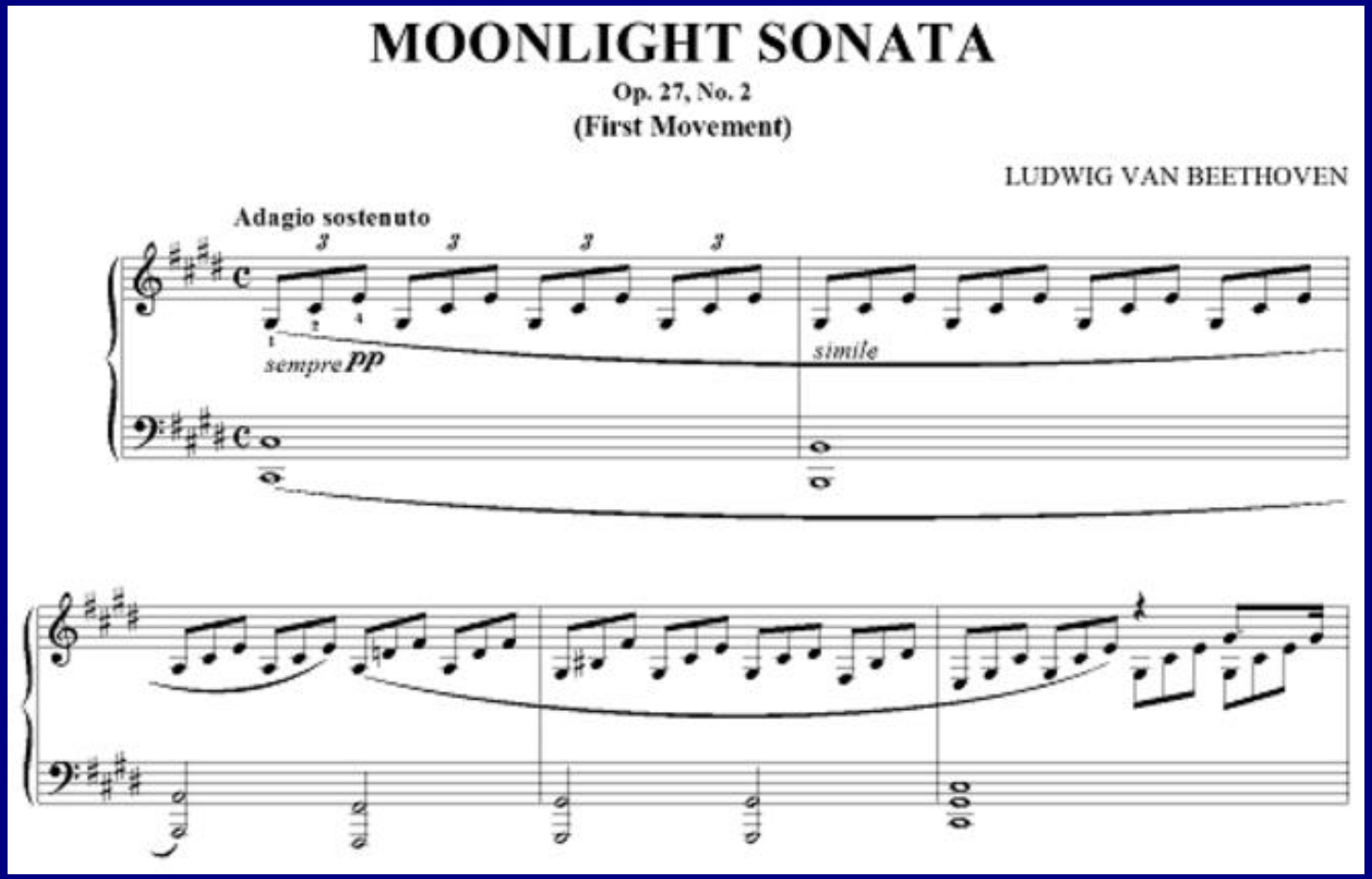
A musical score is just a graph of pitch *versus* time!

Symbiosis: Musing on a Graph



A new composition: ***The Glow-Ball Gavotte.***

Beethoven's *Moonlight Sonata*

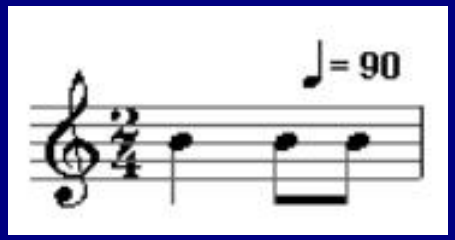


A combination of precision and vagueness.

Precision and Vagueness

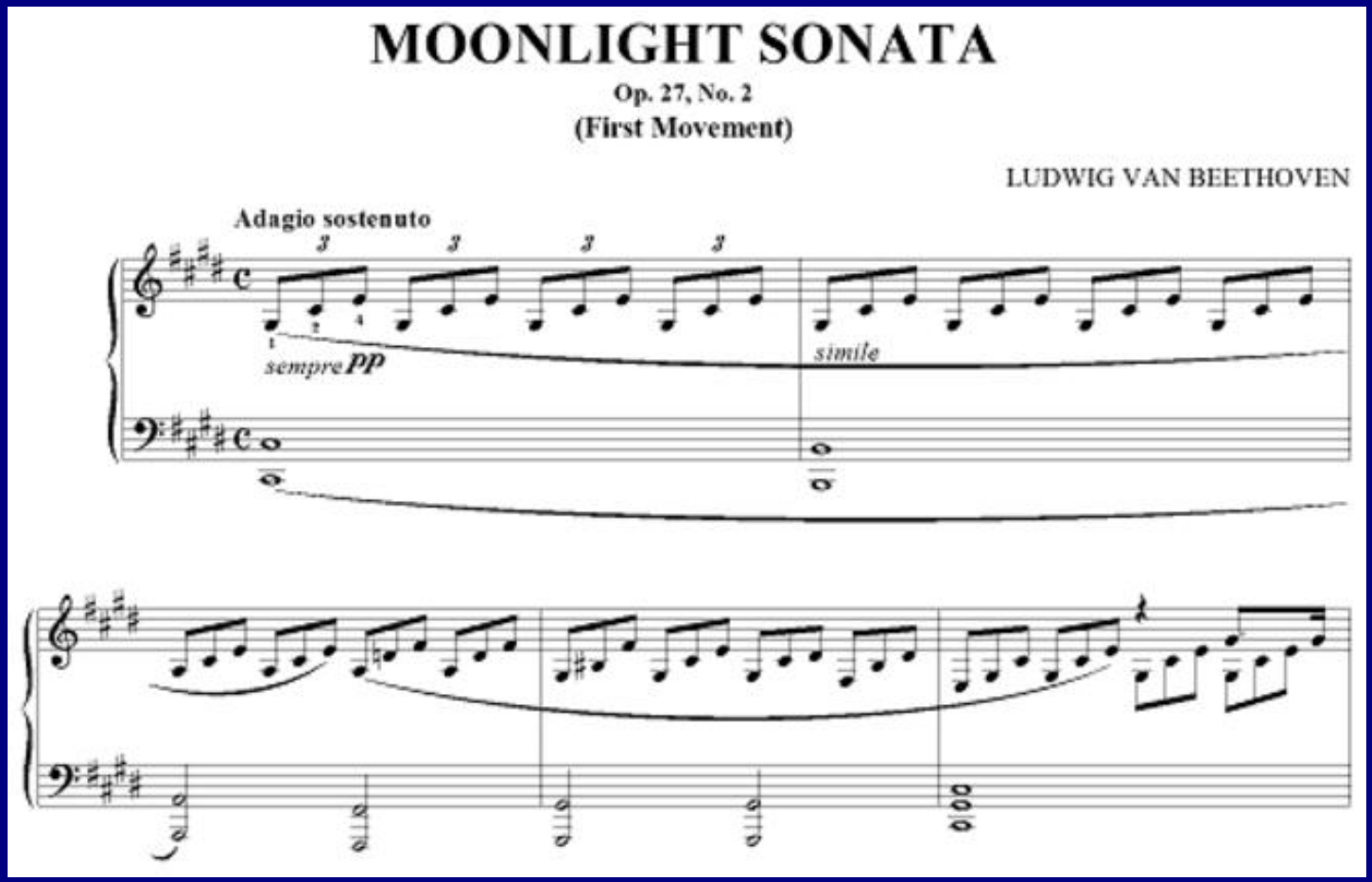
The standard **pitch** is A_4,
the A above Middle C.
Its frequency is **440 Hz.**

By contrast, the **time-scale** is
described only by the phrase
Adagio sostenuto.
It could be given like this:



The **loudness** is specified simply by *sempre* **pp**.

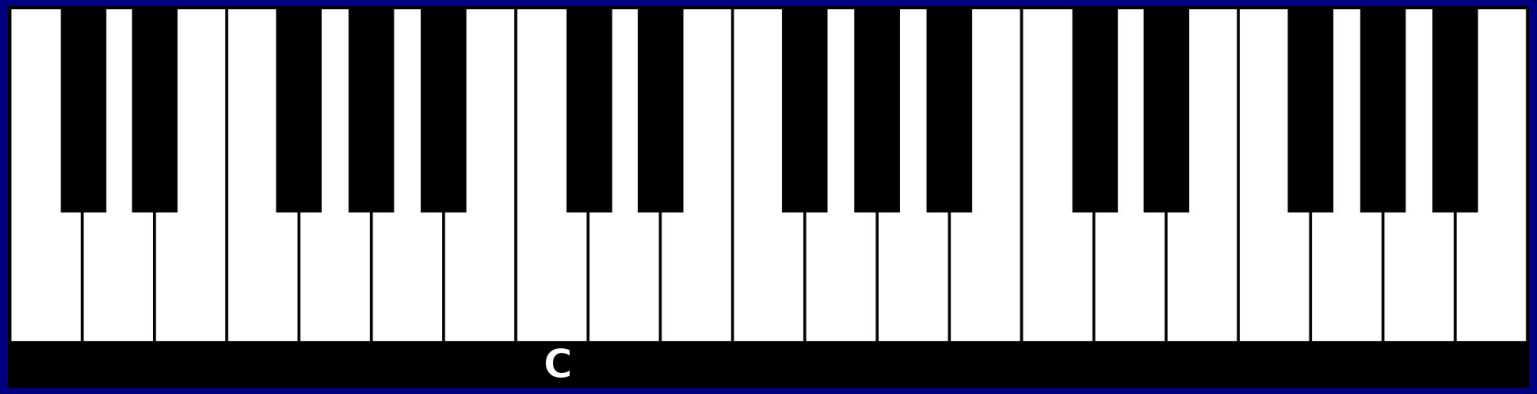
Beethoven's *Moonlight Sonata*



An ingenious and delightful combination of precision (pitch) and vagueness (pp).

Outline

The Piano Keyboard



Where do all the notes come from?

Middle C

Middle C is the 'central note' on the piano.
It is commonly pitched at 261.63 Hz.

The standard frequency of the note A4 is 440 Hz.

$$261.63 = 440 \div 2^{9/12}$$

Where does the peculiar factor $2^{9/12}$ come from?
It arises from the well-tempered scale.

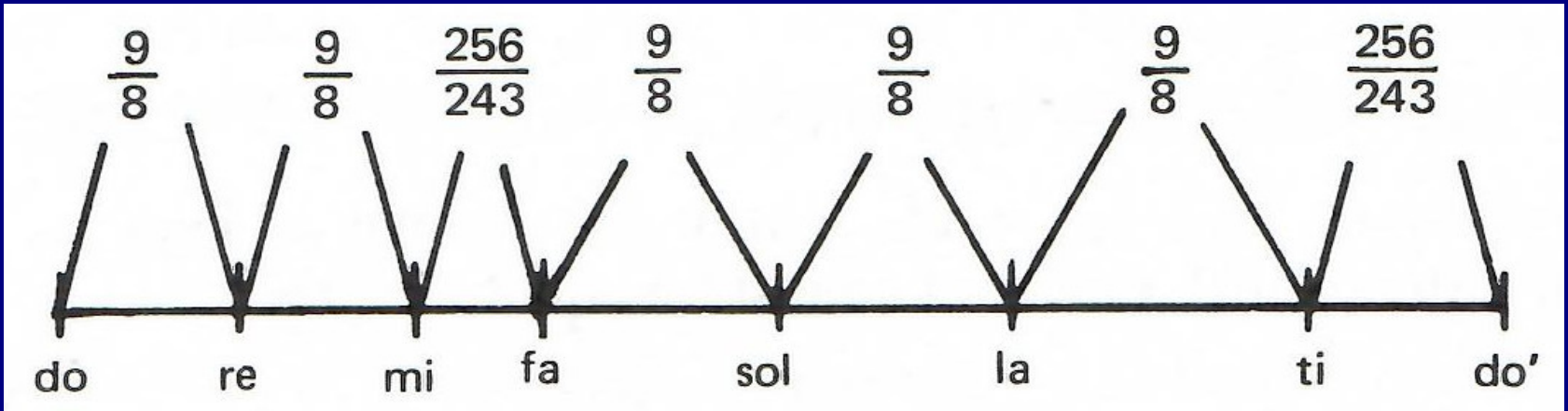
Pythagorean Tuning

Pythagoras discovered that a perfect fifth — with frequency ratio 3:2 — is especially harmonious.

The entire musical scale can be constructed using only the ratios 2:1 (octaves) and 3:2 (fifths).

In the tonic sol-fa scale the eight notes of the major scale are Do, Re, Mi, Fa, So, La, Ti, Do.

Pythagorean Tuning



Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

The Pythagorean Comma

The Pythagoreans noticed that $2^{19} \approx 3^{12}$.

Going up twelve fifths, with ratio $(3/2)^{12}$, and down seven octaves, with ratio $(1/2)^7$ gets us back (almost) to our starting point.

The number

$$3^{12}/2^{19} \approx 1.01364$$

is called the Pythagorean comma.

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is called the Pythagorean comma.

The 13th note is not quite the same as the starting note. The two notes are enharmonics:

$$\text{F}\sharp \quad \neq \quad \text{G}\flat$$

Triads and Just Intonation

The triad — three notes separated by 4 and 3 semitones, such as C–E–G — is of central importance in western music.

In the tuning scheme of Pythagoras, the third (C–E) has a frequency ratio

$$\left(\frac{9}{8}\right)^2 = \frac{81}{64}$$

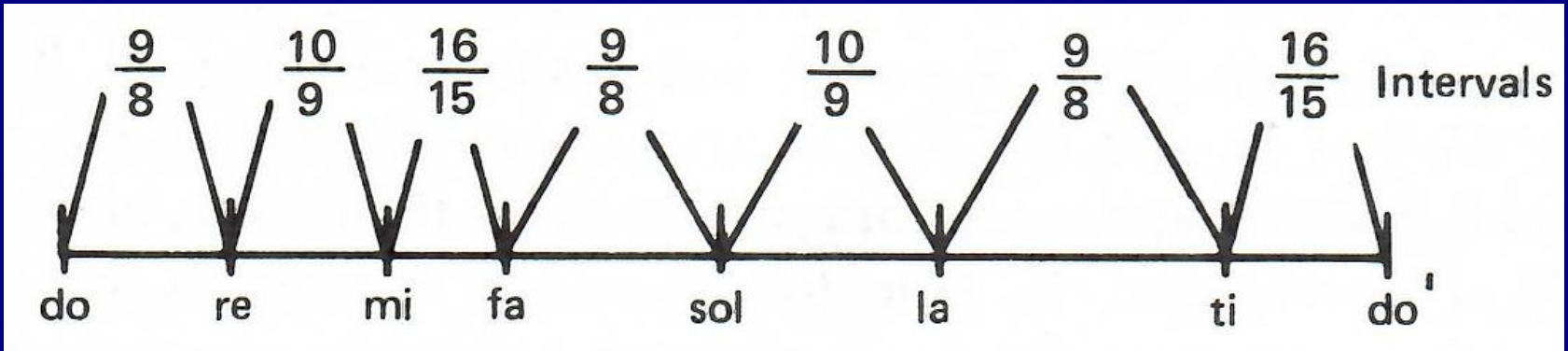
Substituting

$$\frac{81}{64} \approx \frac{80}{64} = \frac{5}{4}$$

the three notes of the triad C–E–G are in the ratio

$$4 : 5 : 6$$

Just Intonation



Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Pythagorean and Just Intonation

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

Pythagorean intonation.

Do	Re	Mi	Fa	So	La	Ti	Do
1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1

Just intonation.

The Well-Tempered Scale

It is impossible to tune a piano so that all fifths have perfect frequency ratios of 3:2.

Idea: Make all semitone intervals equal.

The Well-Tempered Scale

An octave has ratio 2:1. We need a number that yields 2 when multiplied by itself 12 times:

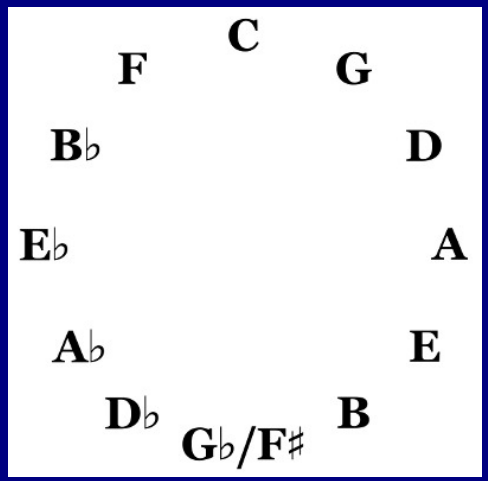
$$\sqrt[12]{2} \approx 1.059$$

In the tempered scale, all intervals are imperfect, but are close enough to be acceptable to the ear.

$\star \qquad \star \qquad \star$

Johann Sebastian Bach's Well-Tempered Clavier is a collection of preludes and fugues in all 24 keys.

Organizing Scheme: the Circle of Fifths



The Circle of Fifths represents the relationship between musical pitch and key signature.

It shows the twelve tones of the chromatic scale.

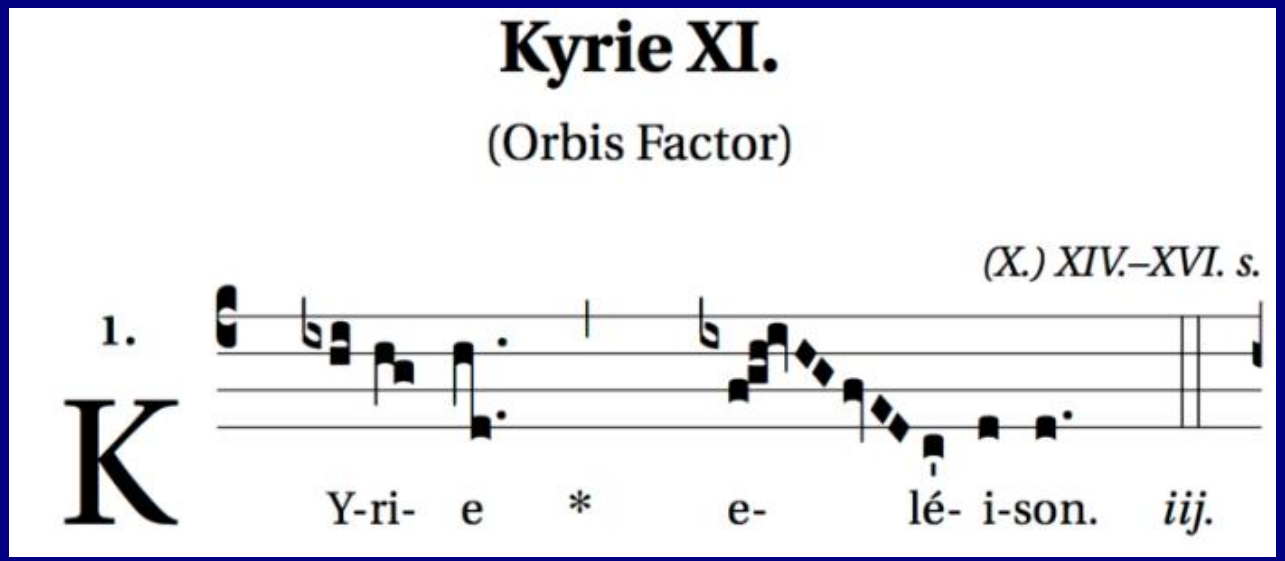
The Circle is useful in harmonising melodies and building chords.

Outline

The Geometry of Canons

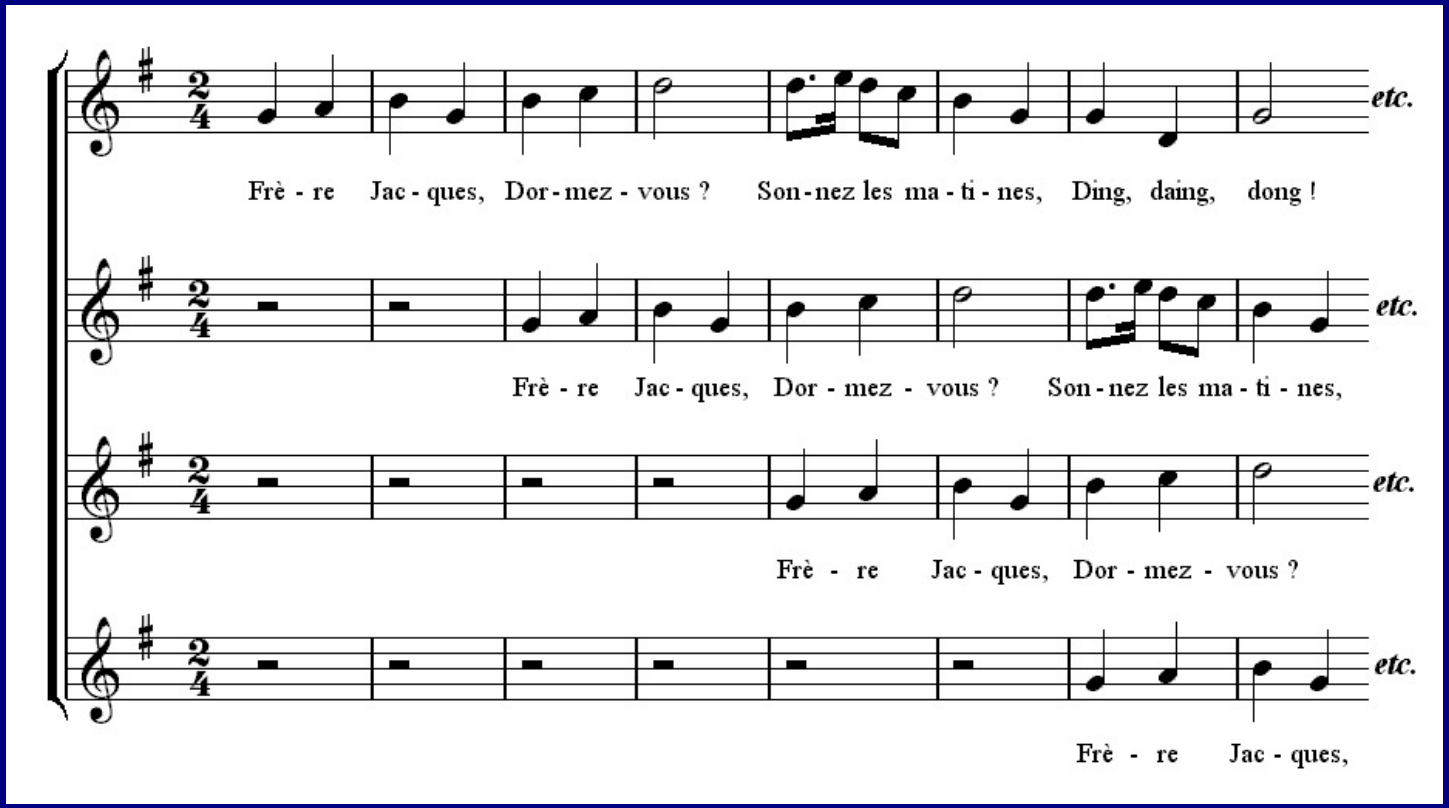
Gregorian chant is a monophonic, unaccompanied chant, developed during the 9th and 10th centuries.

There are no fixed measures and no time signature.



Gradually, regular division into bars or measures, each of the same fixed length, emerged.

The Geometry of Canons



A manuscript at the French National Library contains 86 canons by J.-P. Rameau, including Frère Jacques.

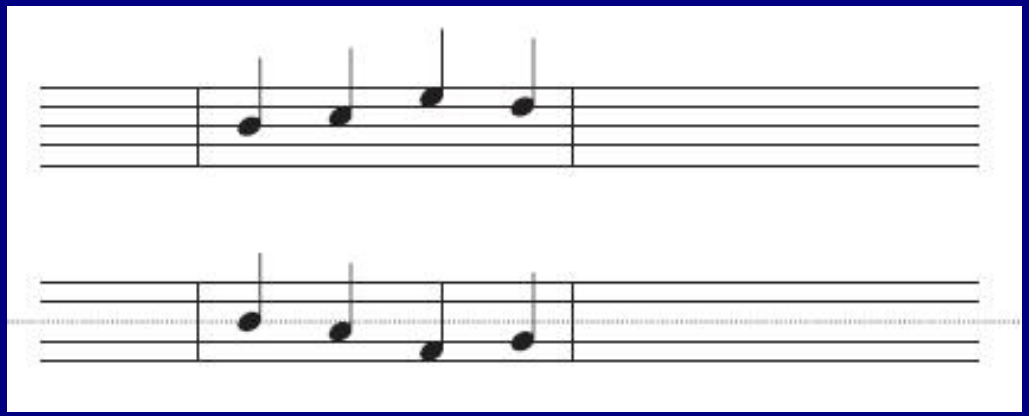
The Geometry of Canons

- Canon at the Unison
- Canon at an Interval
- Canon *contrario motu*
- Retrograde Canon
- Perpetual Canon

Johann Sebastian Bach was the grand master of canon form.

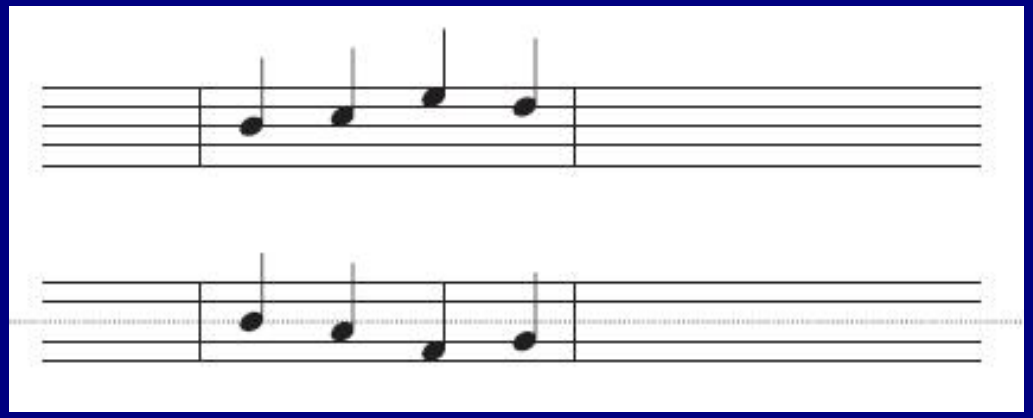
The transformations used in Canon Form are described by mathematical groups.

Simple Transformations

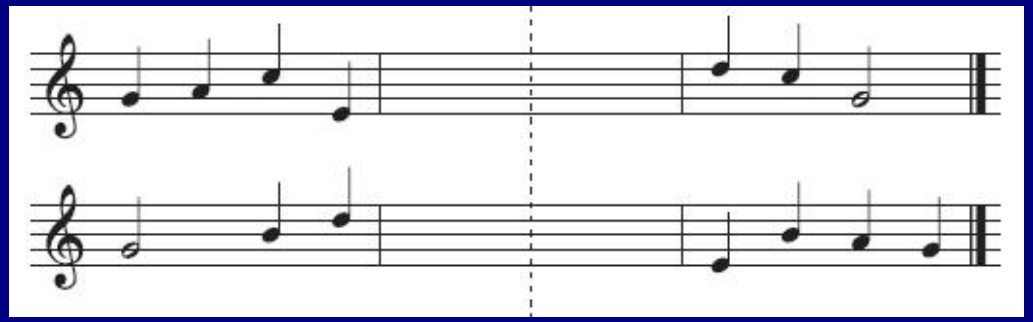


Inversion, or rotation about a horizontal line.

Simple Transformations



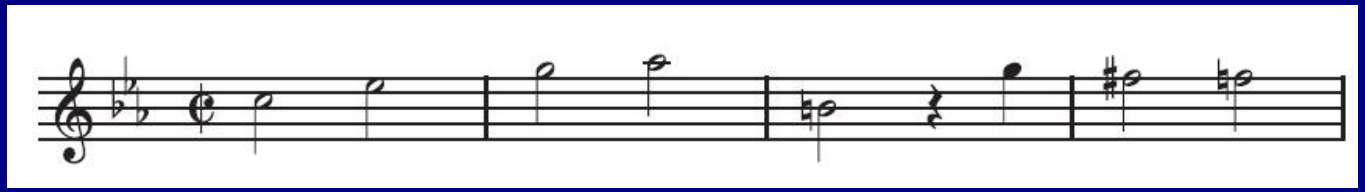
Inversion, or rotation about a horizontal line.



Retrogression, or rotation about a vertical line.

“A Musical Offering”

Frederick the Great provided Bach with a theme:



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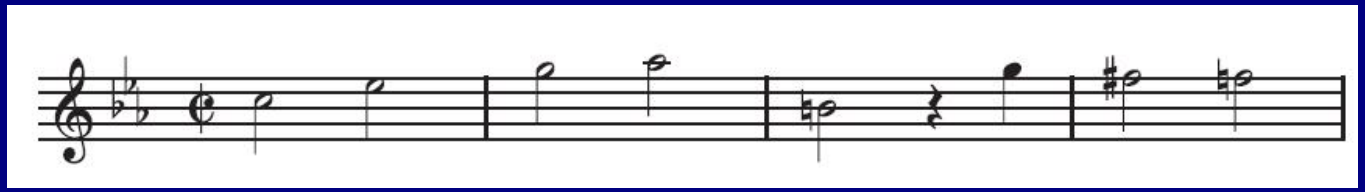
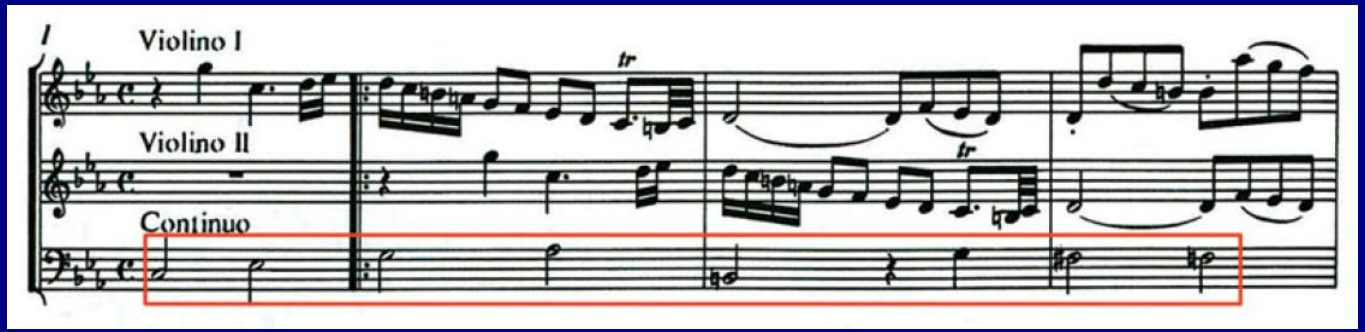


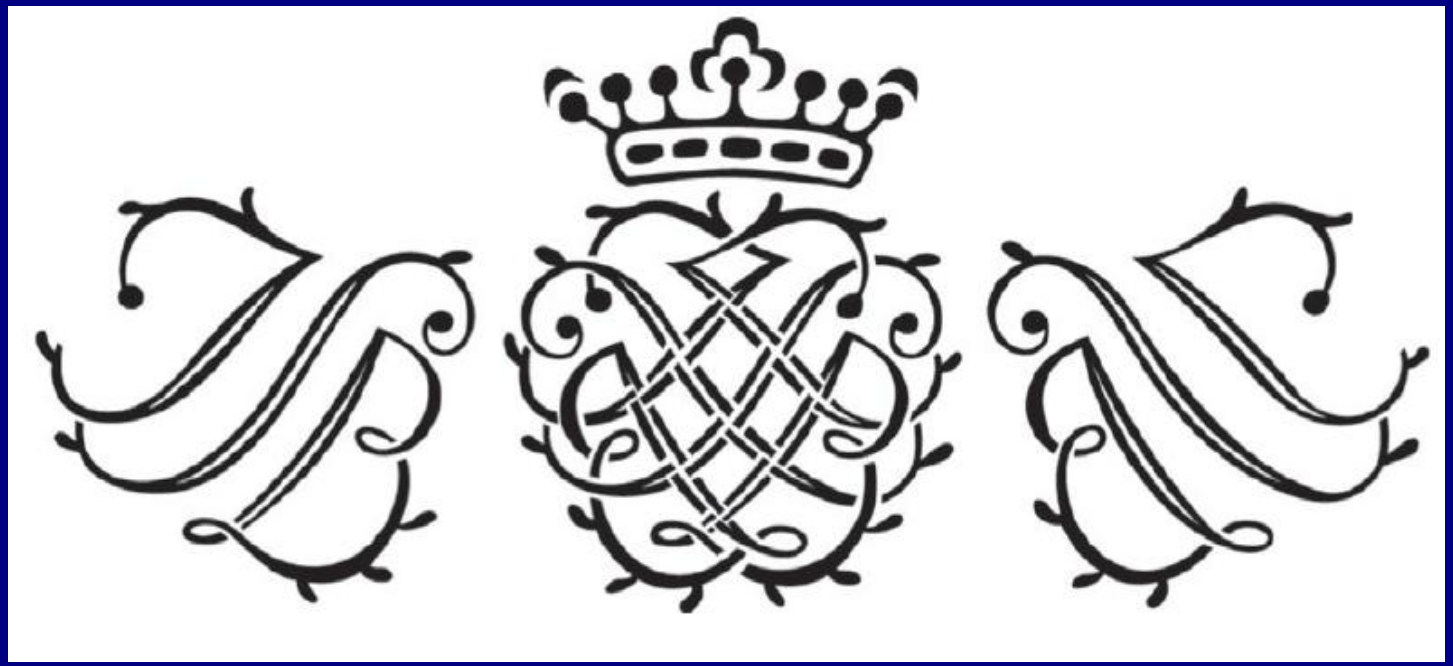
Bach worked this into *A Musical Offering*, a collection of ten canons, a sonata and two fugues.



The work has several riddles and hidden jokes.

A “Graphical” Offering

Bach was deeply familiar with symmetry.



This is illustrated by the seal he designed in 1722.

Outline

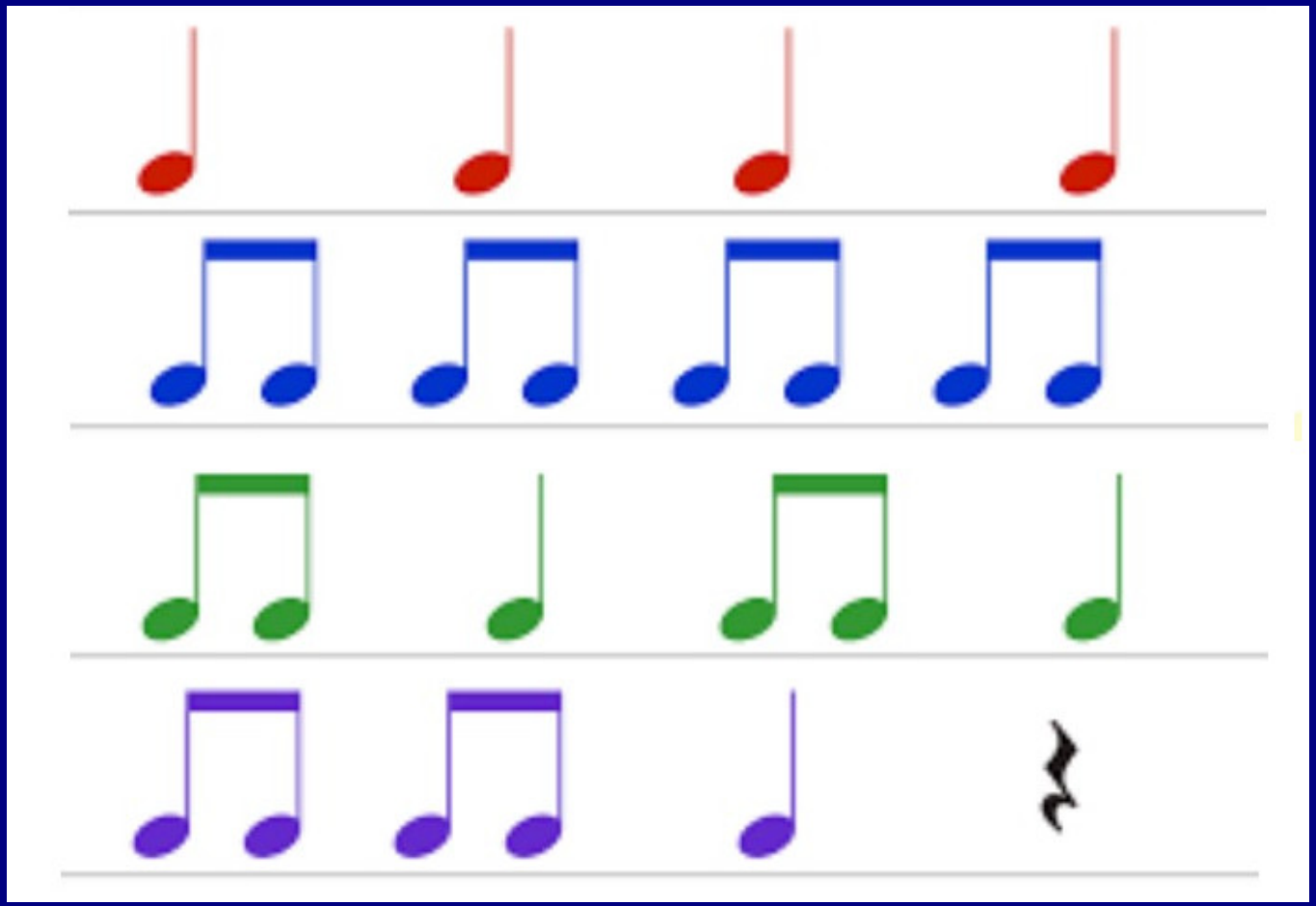
Leibniz on Music

Gottfried Wilhelm Leibniz

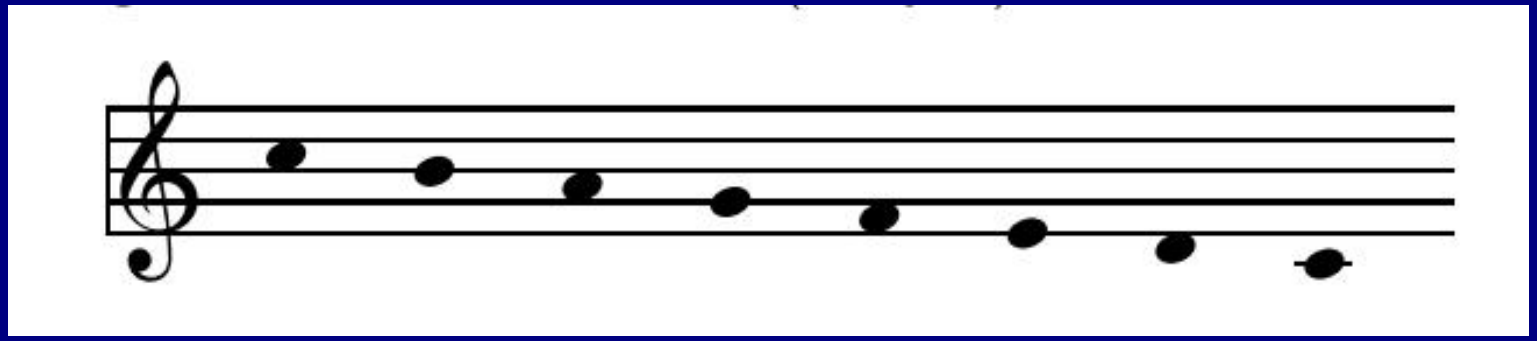
(1646–1716)

"Music is the pleasure the human mind experiences from counting without being aware that it is counting."

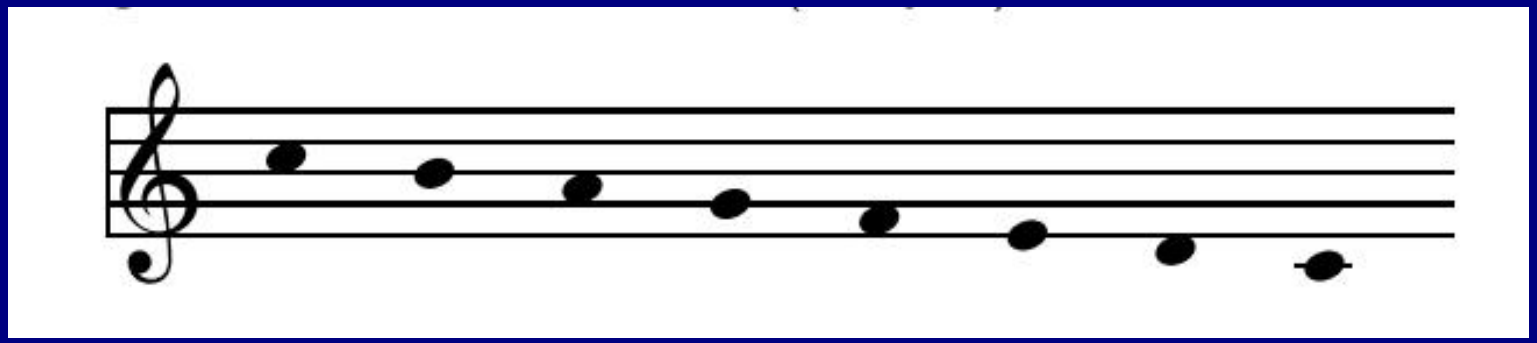
Beating the Time

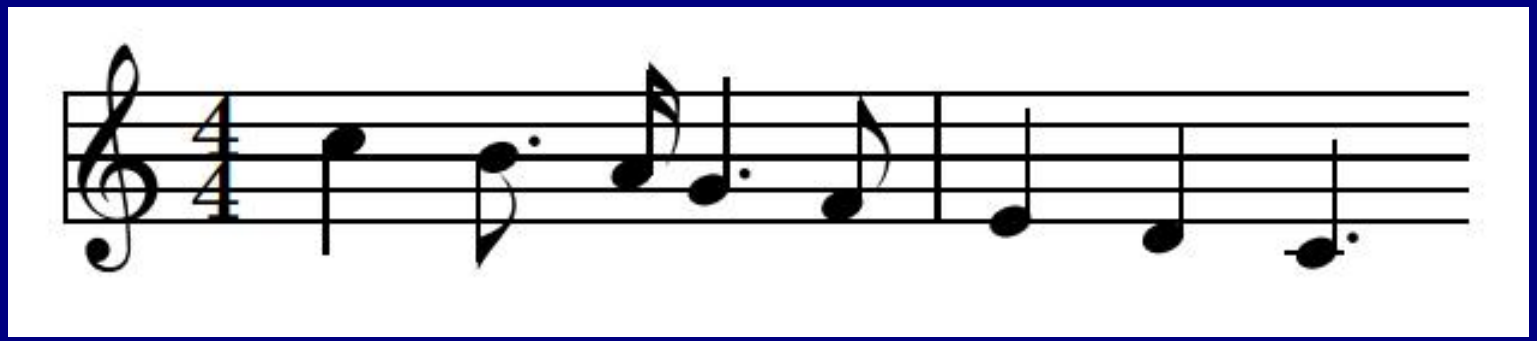


Rhythm in Music



Rhythm in Music

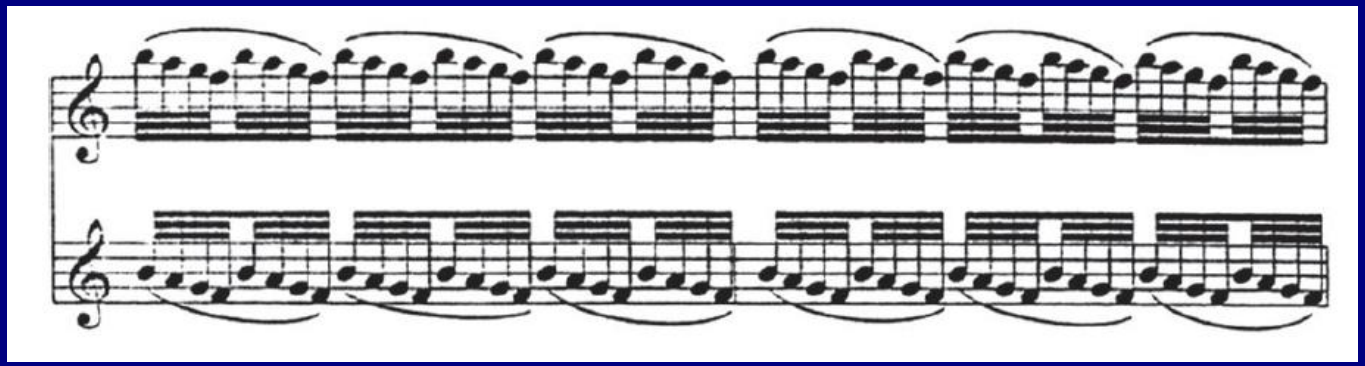




Joy to the World.

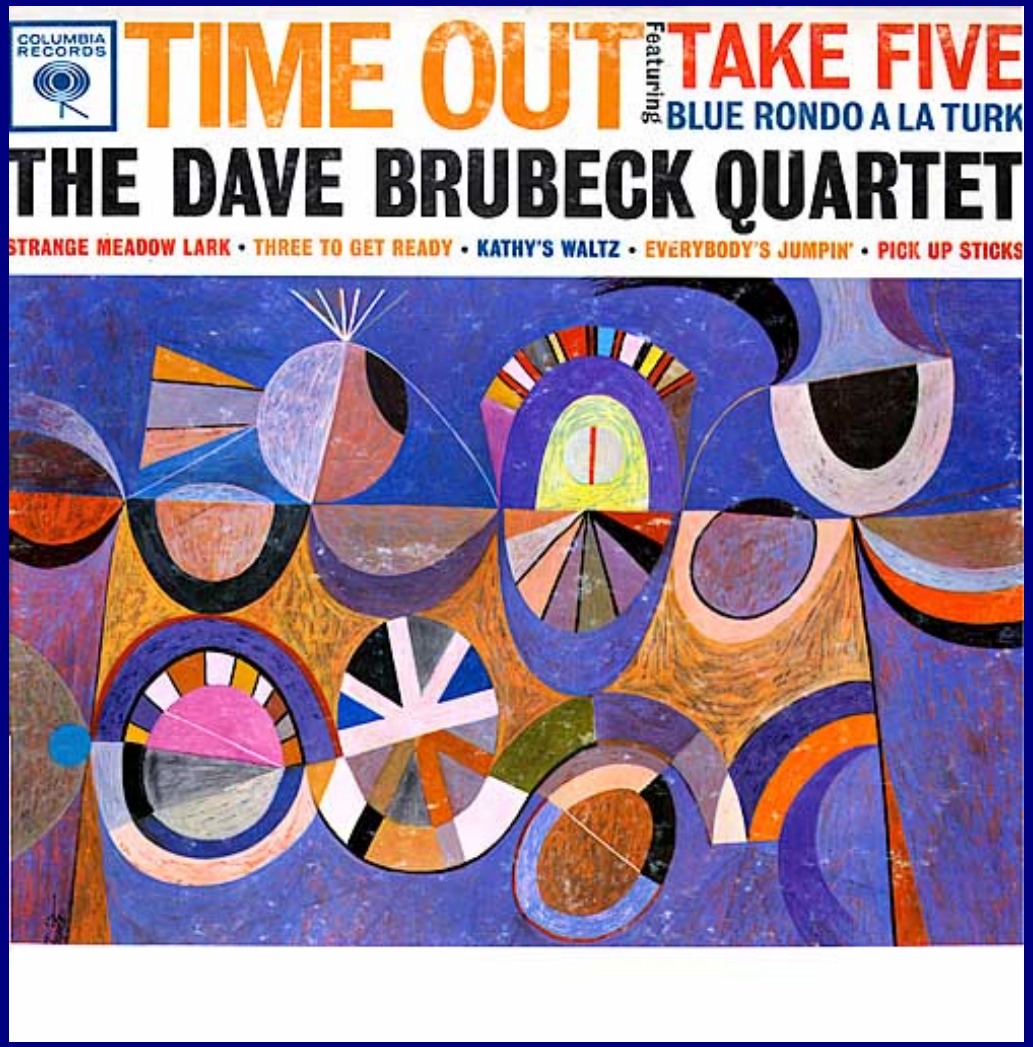
Repetition in Music

Opening bars of Beethoven's *Moonlight Sonata.*

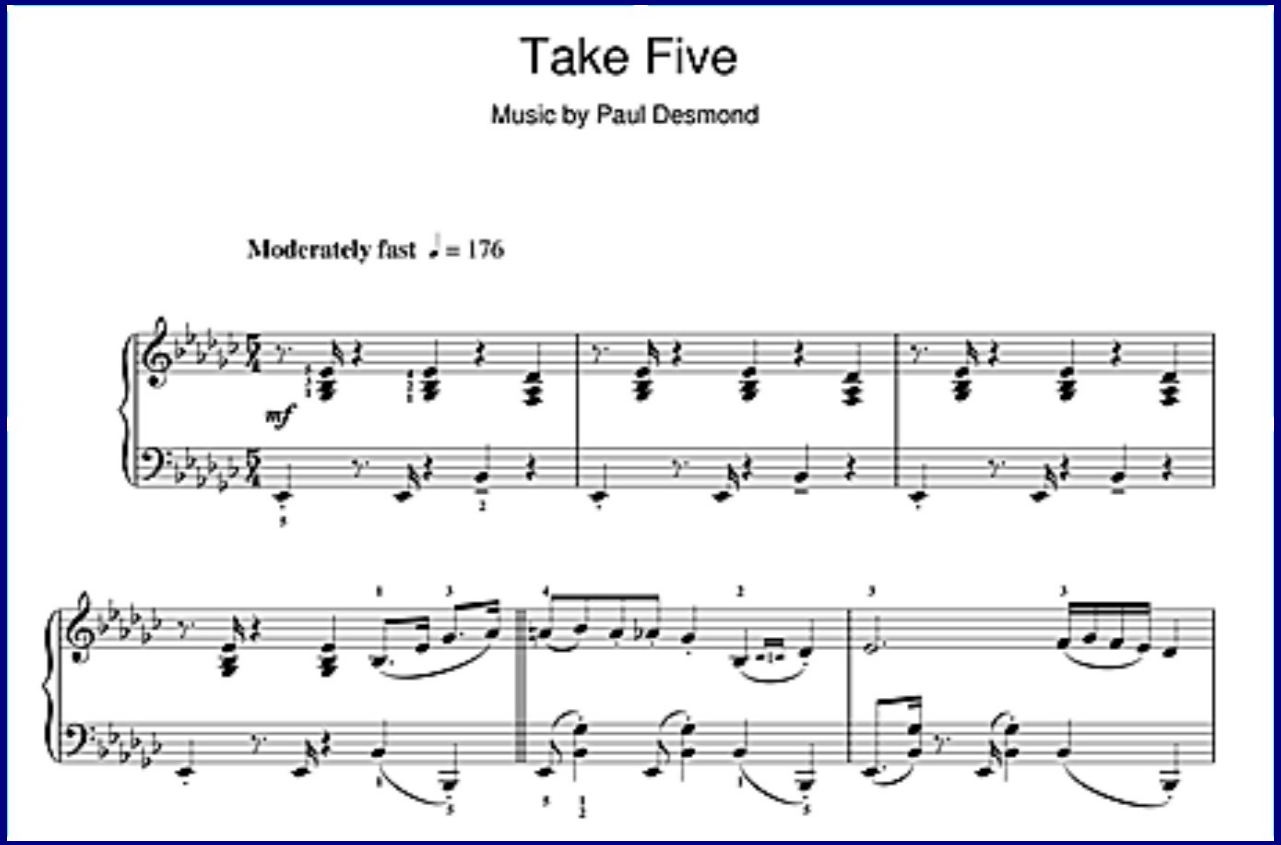


Claude Debussy's *La Mer* (2nd movement, bar 72).

Time Out (1959)

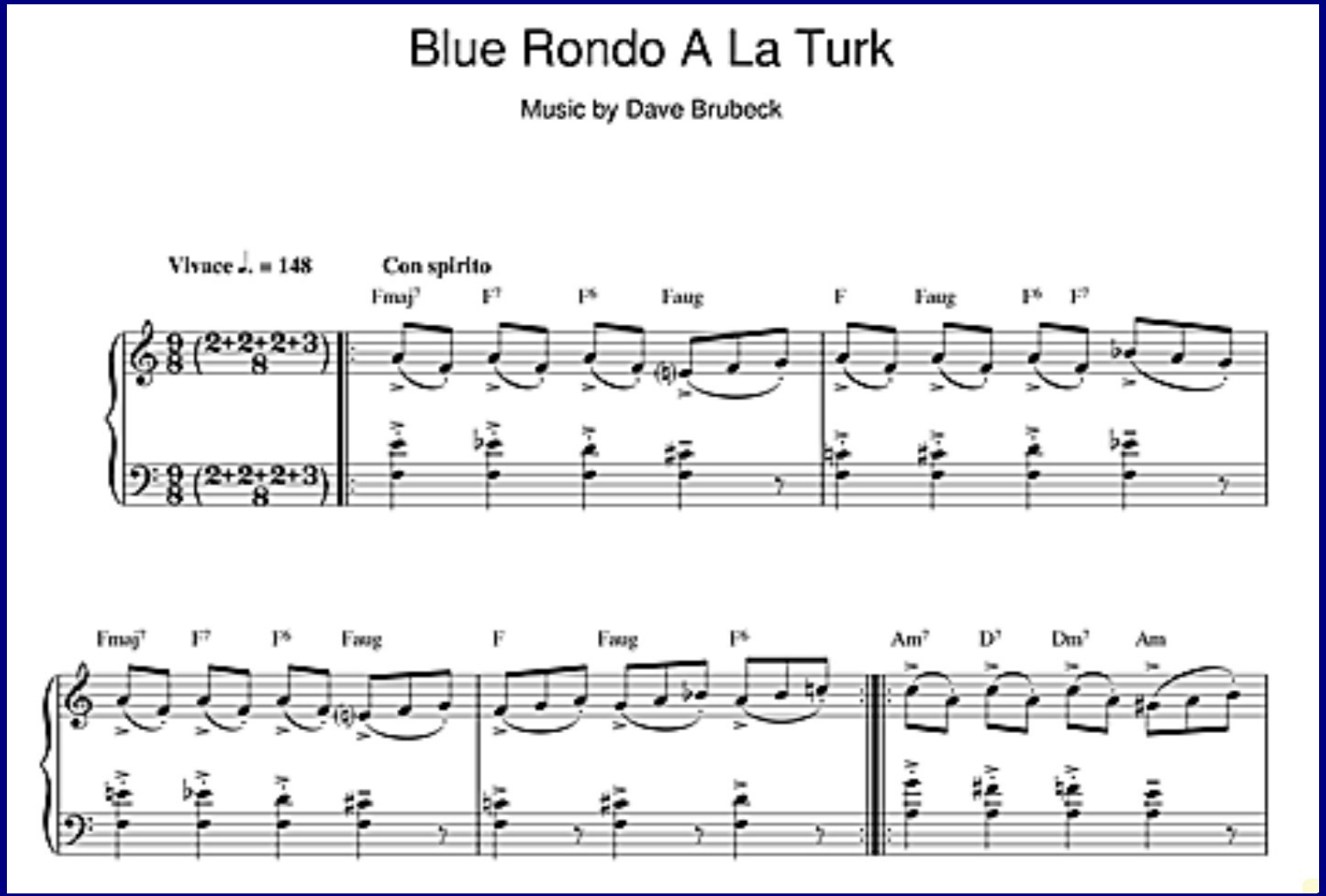


Take Five: 5/4 Time

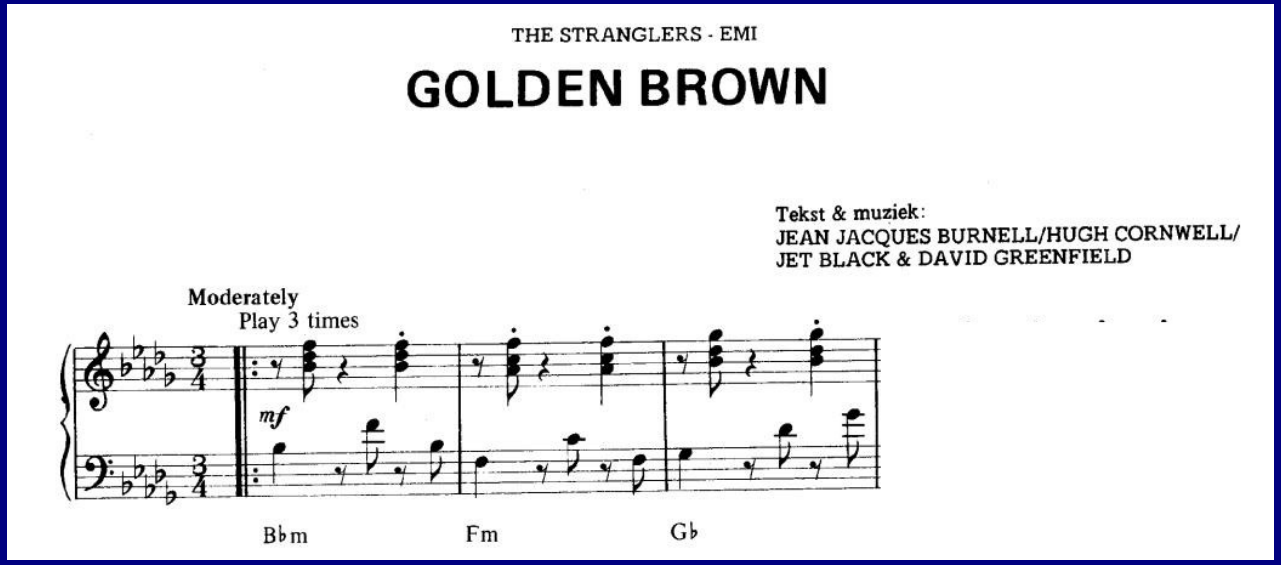


Based upon use of time signatures that were unusual for jazz: $\frac{5}{4}$, $\frac{9}{8}$, etc.

Blue Rondo a la Turk: 9/8 Time



The Stranglers: *Golden Brown*


THE STRANGLERS - EMI

GOLDEN BROWN

Tekst & muziek:
JEAN JACQUES BURNELL/HUGH CORNWELL/
JET BLACK & DAVID GREENFIELD

Moderately
Play 3 times

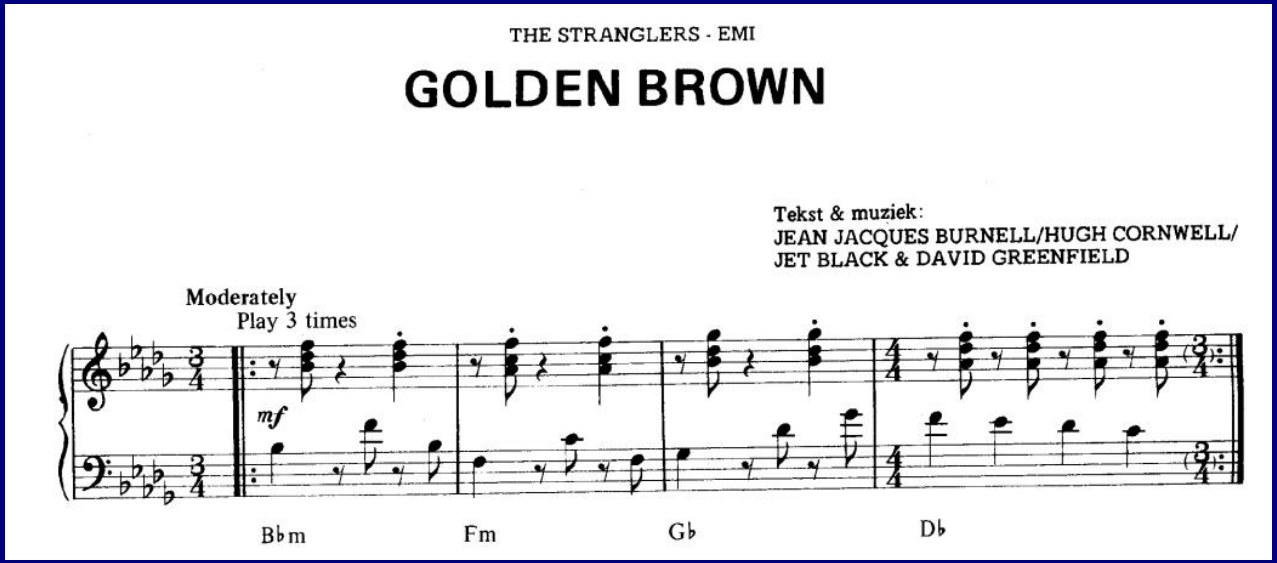
mf

B♭m Fm G♭

Can you work out the time signature?

Listen to the video clip.

The Stranglers: *Golden Brown*



It looks like (3+3+3+4)/8 = 13/8.

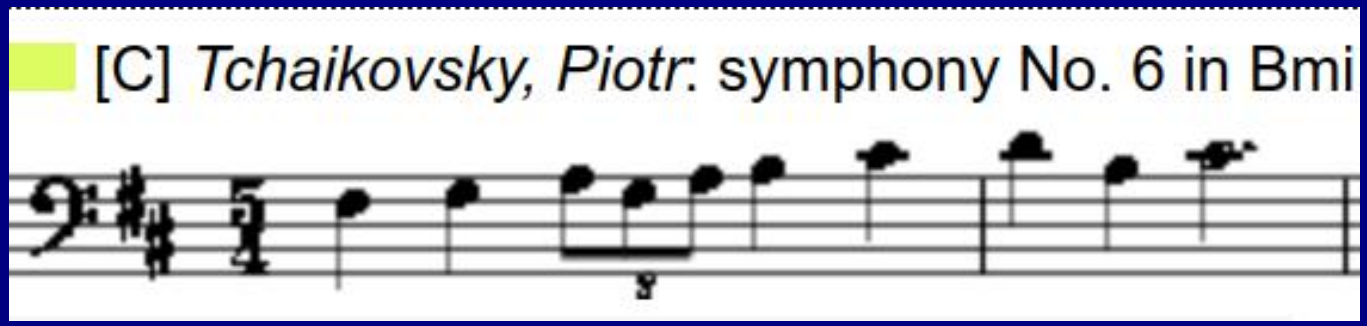
Listen to the video clip.

Pink Floyd: *Money*

Pink Floyd: *Money*

***Money* is composed mainly in 7/4 time.**

Tchaikovsky's *Pathétique Symphony*



The second movement, a dance form in 5/4 time, has been described as a "limping waltz".

Outline

Ubiquity and Beauty of Symmetry

Symmetry is all around us.

- Many buildings are symmetric.
- Our bodies have bilateral symmetry.
- Crystals have great symmetry.
- Viruses can display stunning symmetries.
- At the sub-atomic scale, symmetry reigns.
- Galaxies have many symmetries.

The Taj Mahal

A Face with Symmetry: Halle Berry

Halle Berry

Berry Halle

An Asymmetric Face: You know Who!

Symmetry and Group Theory

Symmetry is an essentially geometric concept.

The mathematical theory of symmetry is algebraic.
The key concept is that of a group.

A group is a set of elements such that any two elements can be combined to produce another.

Instead of giving the mathematical definition,
I give an example to make things clear.

The Klein 4-Group

The four orientations of a book can be described in terms of four simple rotations:

- **P:** Place book upright with front cover upright
- **R:** Rotate 180° about vertical through centre
- **I:** Rotate 180° about horizontal through centre
- **RI:** Rotate 180° about perp. through centre

These operations make up the **Klein 4-Group.**

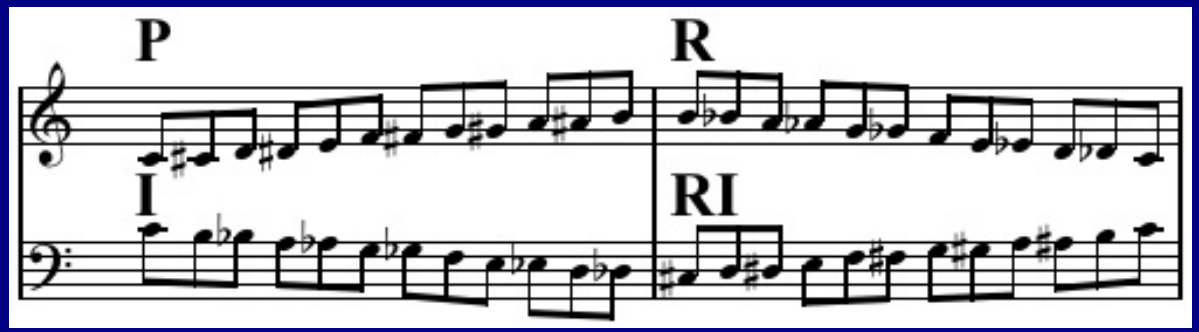
Twelve-tone Music

Table: Klein 4-Group.

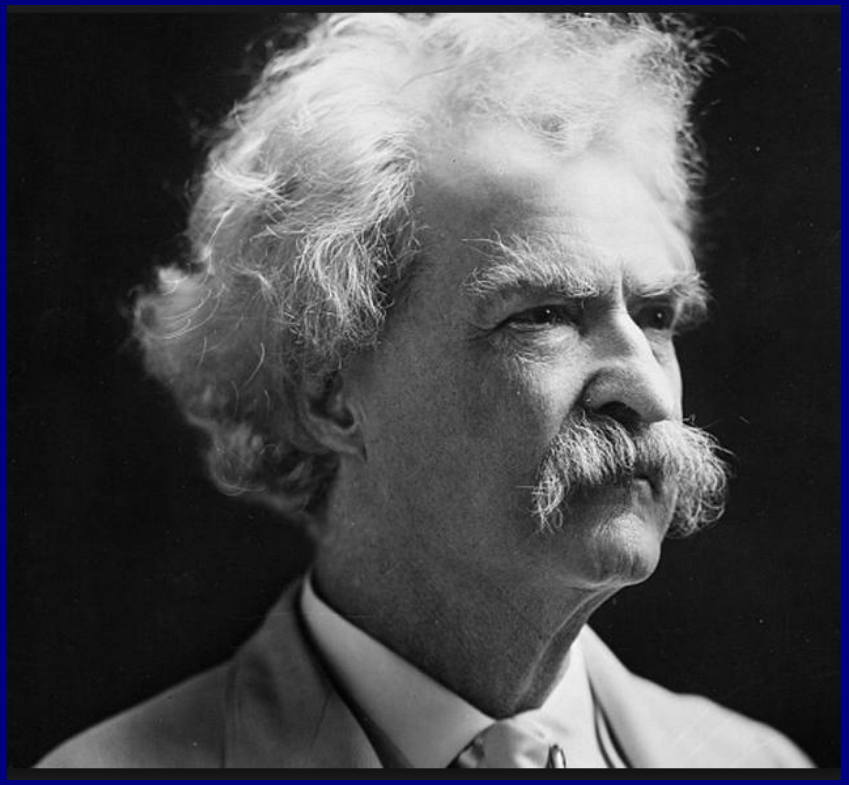
	P	R	I	RI
P	P	R	I	RI
R	R	P	RI	I
I	I	RI	P	R
RI	RI	I	R	P

The Klein 4-group is the basic group of transformations in twelve tone music.

The operations are retrogression (R), inversion (I) and the rotation (RI).



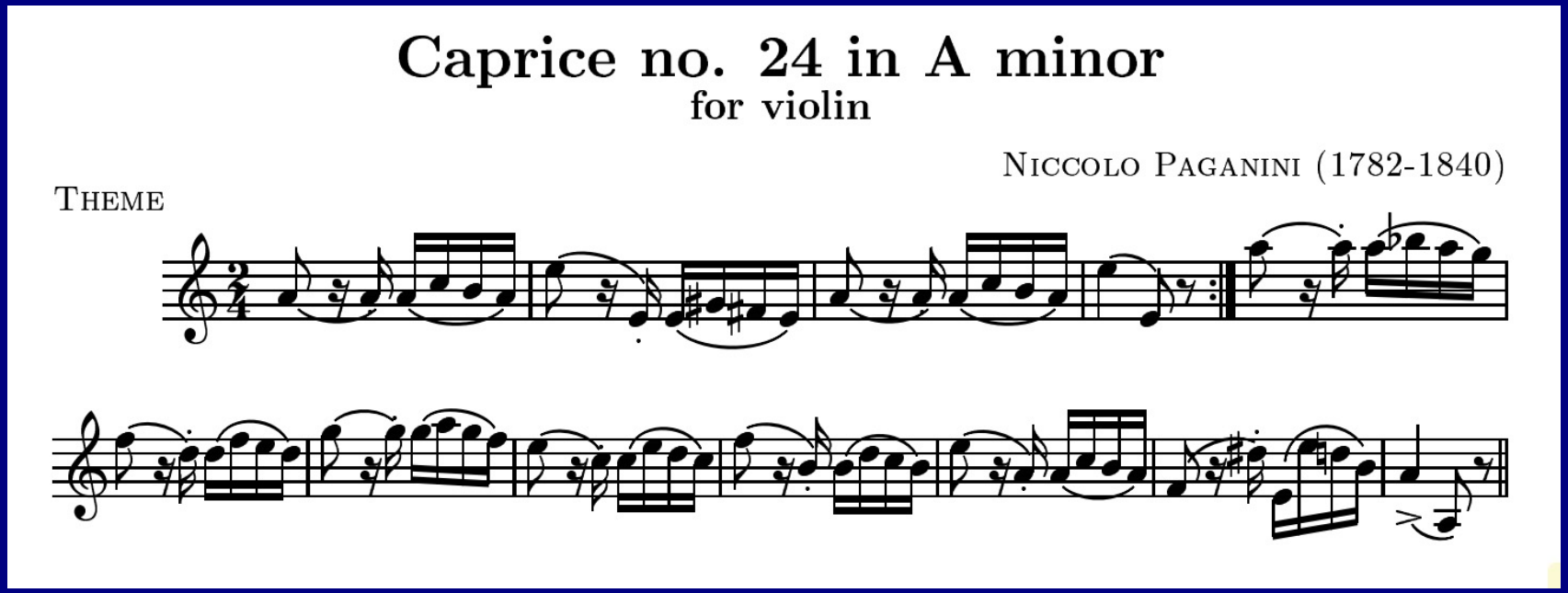
Mark Twain on Wagner



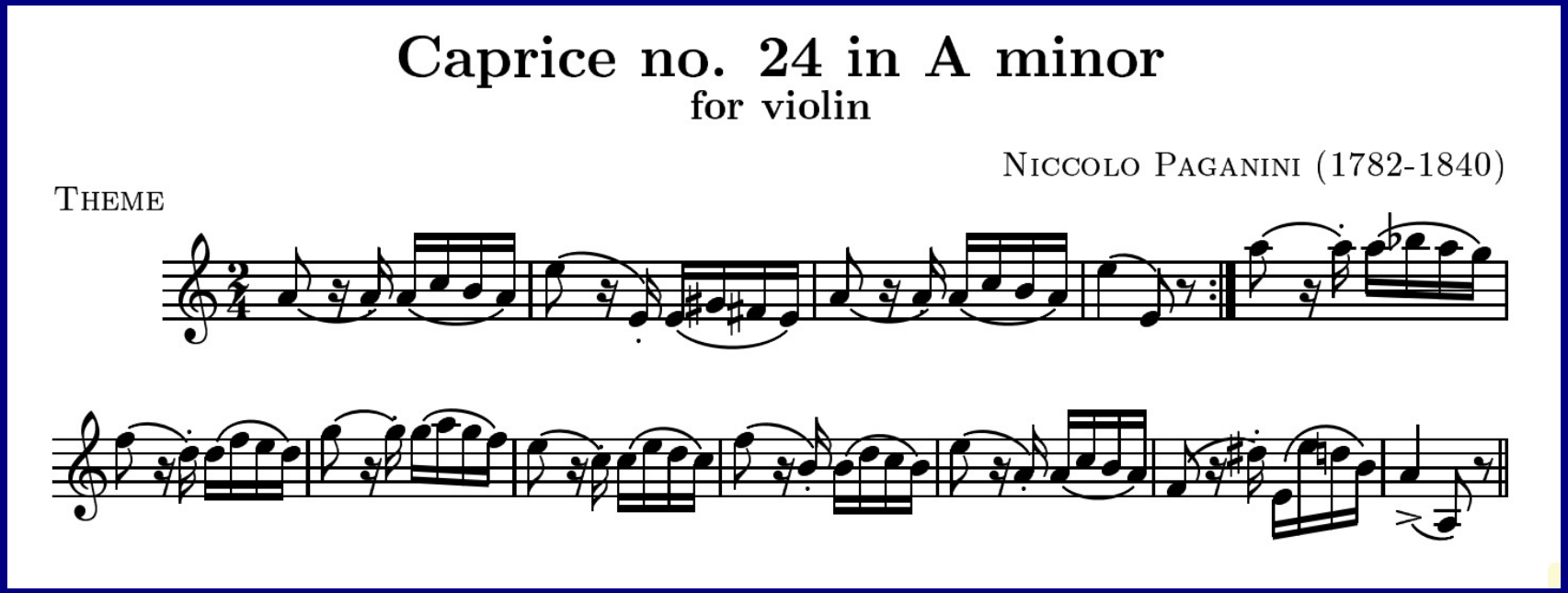
"Wagner's music is much better than it sounds."

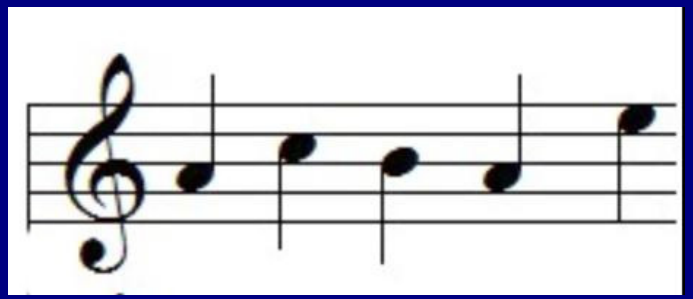
I am not aware whether Mark Twain ever commented on Arnold Schoenberg's music !

Paganini: Caprice 24

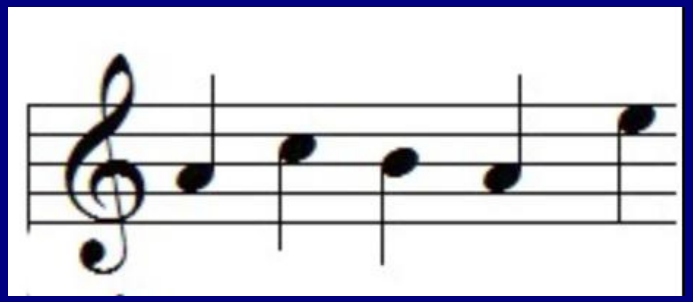


Paganini: Caprice 24



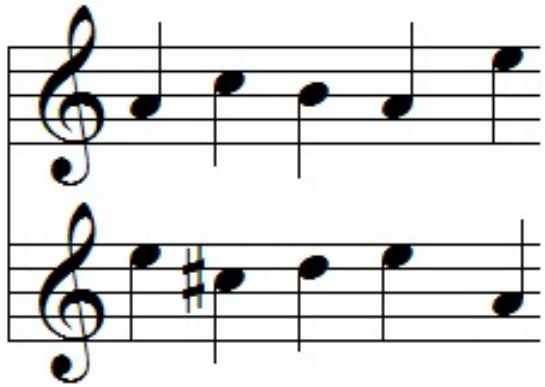


Rachmaninov's *Rhapsody*



Paganini (part of 24th capriccio theme)

Rahcmaninoff (inversion: 18th variation theme)

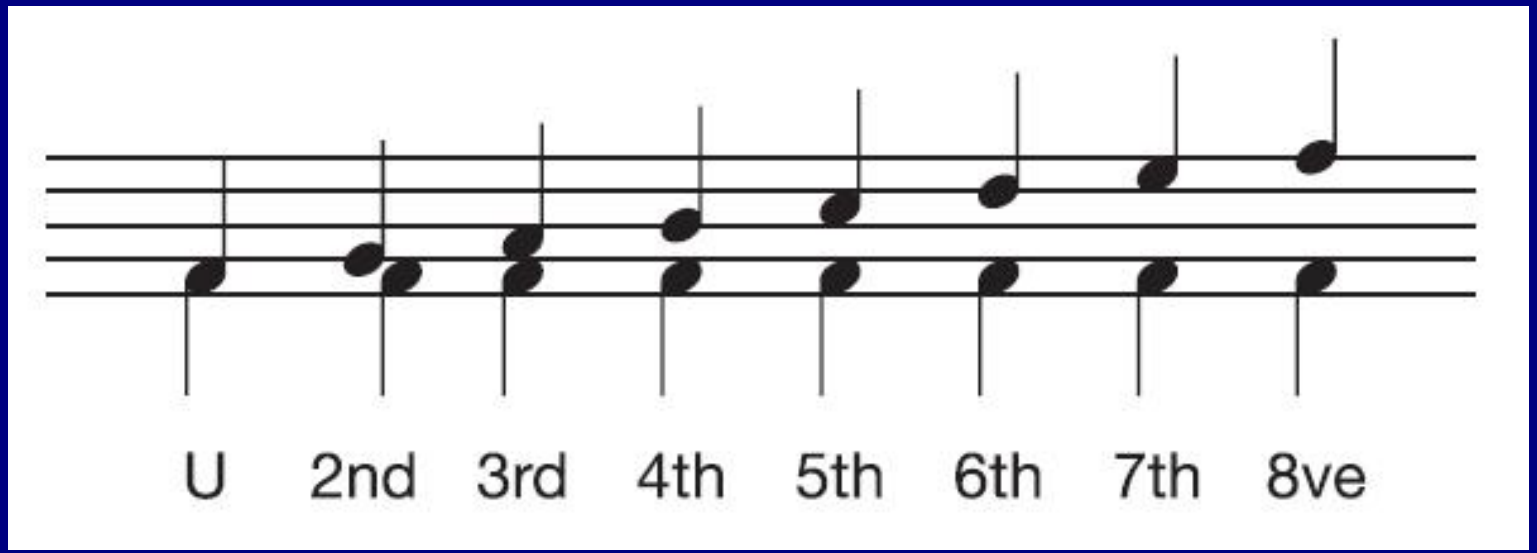


file:///home/peter/Dropbox/Music/RDS-MusicClips/Rachmaninov-Clip1.mp4

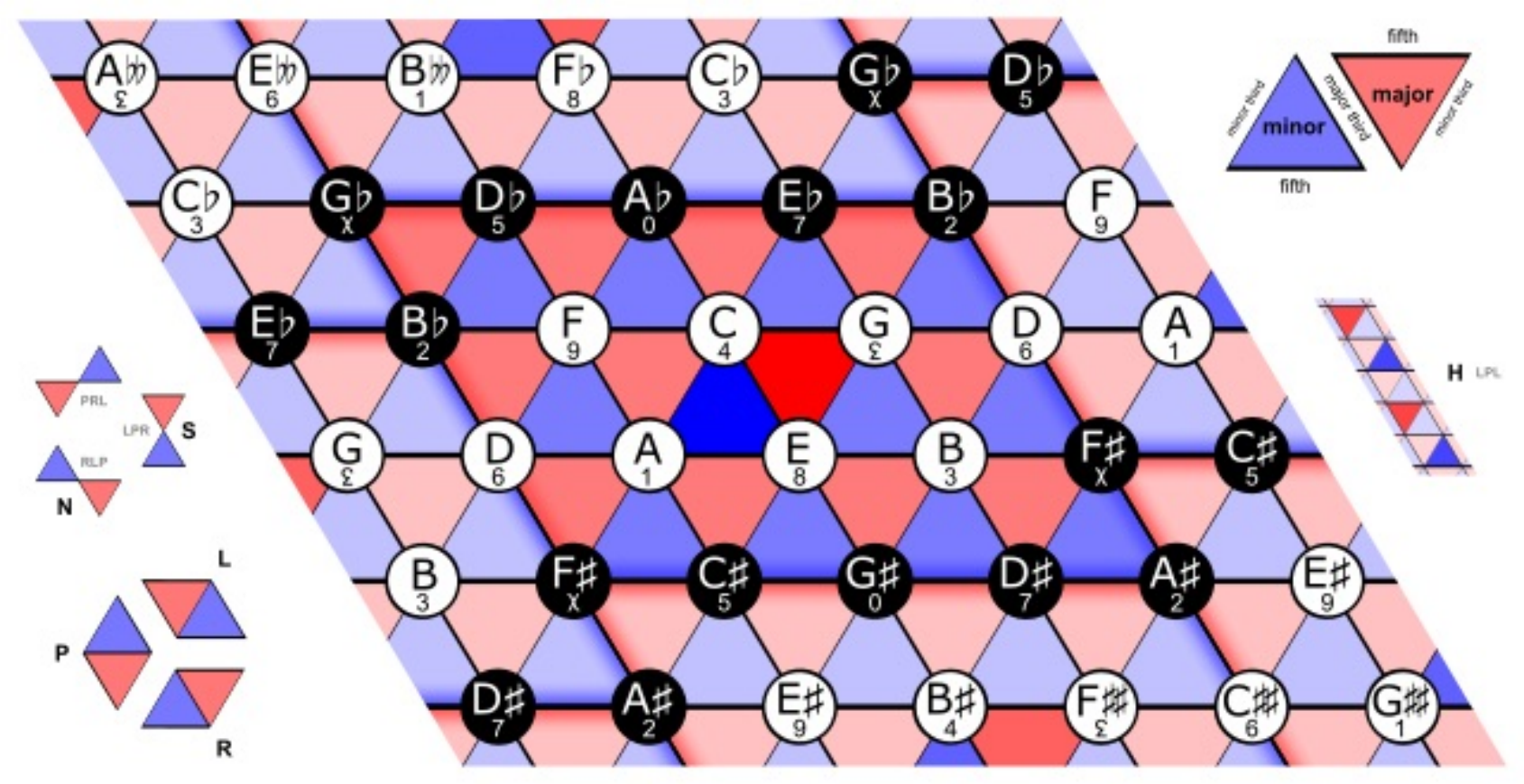
Outline

Musical Intervals. Chords

An Interval is two notes sounded together.
Interval distances are counted inclusively.



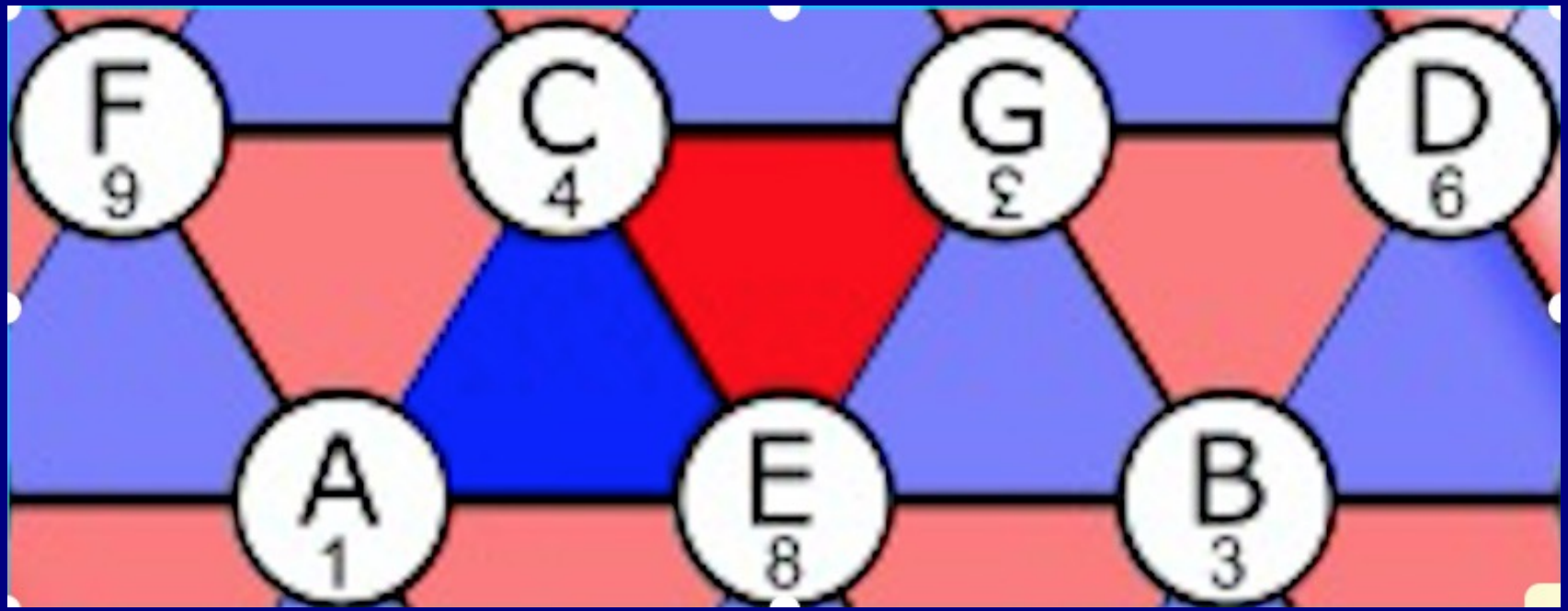
The Tonnetz



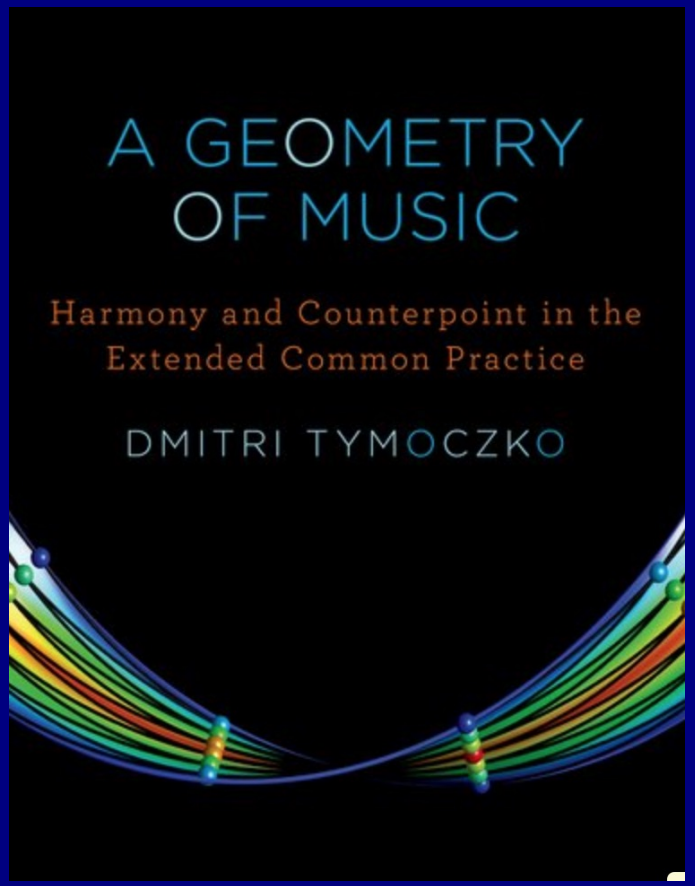
This has the topology of a torus or doughnut.

Animated gif of Tonnetz on a Torus.

The Tonnetz — Detail

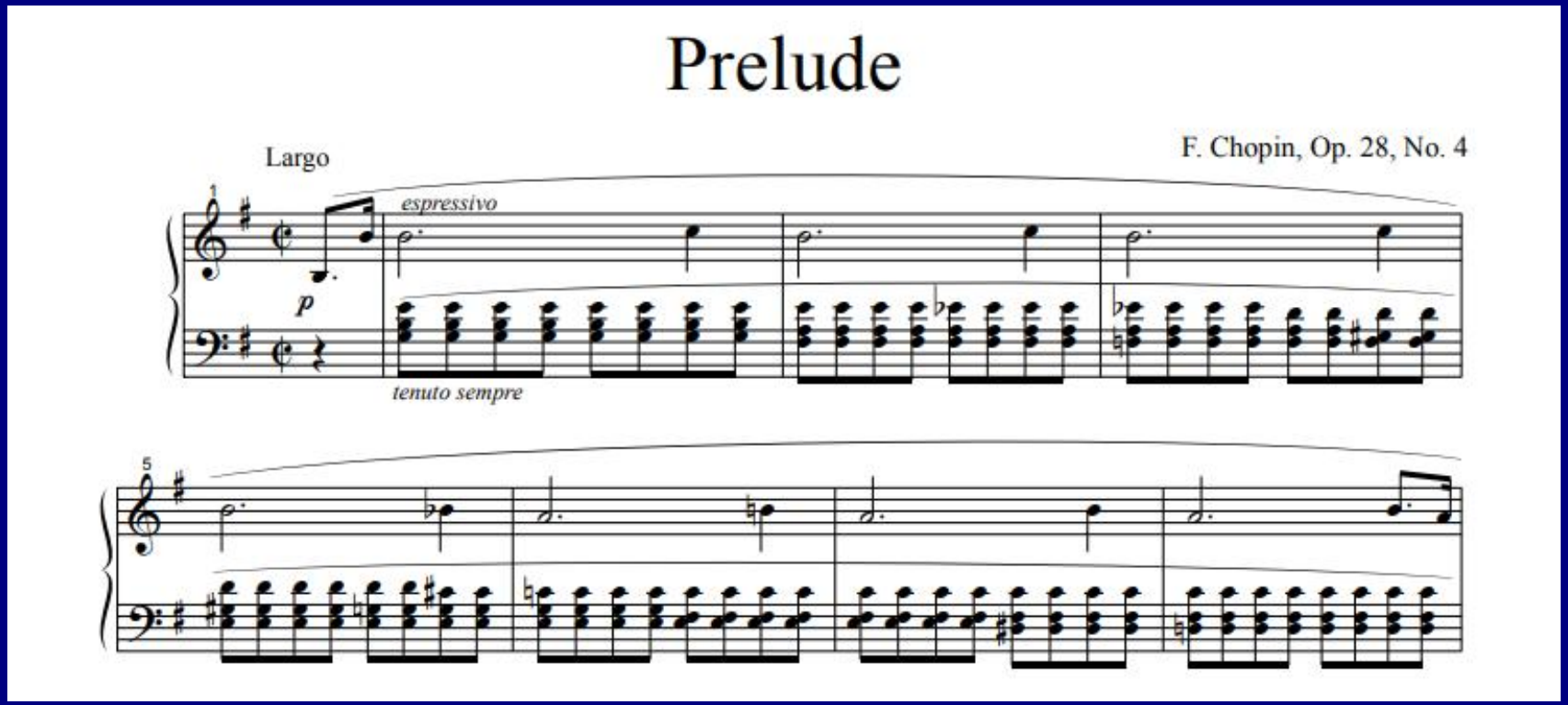


A Geometry of Music

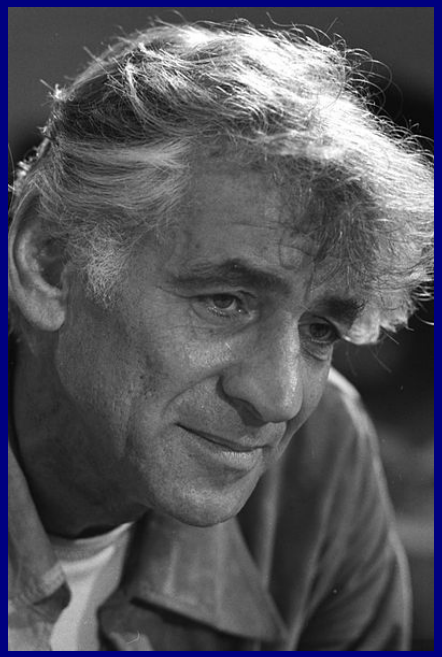


https://vimeo.com/20300784

Chopin's Prelude Op. 28, No. 4



Leonard Bernstein



"Why do so many of us try to explain the beauty of music, apparently depriving it of its mystery?

" . . . music is not only a mysterious and metaphorical art; it is also born of science.

"It is made of mathematically measurable elements.

. . . any explication of music must combine mathematics with aesthetics."

Rachmaninov's *Rhapsody*

file:///home/peter/Dropbox/Music/RDS-MusicClips/Rachmaninov-Clip1.mp4

Thank you