Magnums

Counting Sets with Surreal Numbers

Peter Lynch & Michael Mackey
School of Mathematics & Statistics
University College Dublin

Seminar, G4G, 21 December 2023

Outline

Outline

Magnums and Subsets of $\mathbb{N}$

The aim of this work is to define a number

$$m(A)$$

for subsets A of $\mathbb{N}$ that corresponds to our intuition about the size or magnitude of A.

We call $m(A)$ the magnum of A.

Magnum = Magnitude Number

Magnums and Subsets of $\mathbb{N}$

The aim of this work is to define a number

$$m(A)$$

for subsets A of $\mathbb{N}$ that corresponds to our **intuition** about the size or magnitude of A.

We call $m(A)$ the **magnum of A**.

Magnum = Magnitude Number

"It is by logic that we prove,
but by intuition that we discover."
[Henri Poincaré]

Galileo Galilei (1564–1642)

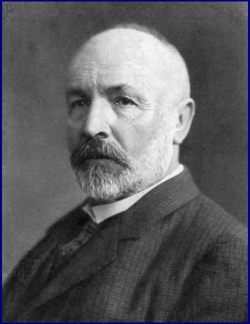
Every number n can be matched with its square n^2.

In a sense, there are as many squares as whole numbers.

1	2	3	4	5	6	7	8	…
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	…
1	4	9	16	25	36	49	64	…

Outline

Georg Cantor (1845–1918)



Cantor discovered many remarkable properties of infinite sets.

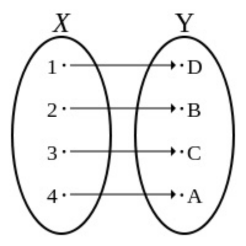
Georg Cantor (1845–1918)

- Invented Set Theory.
- One-to-one Correspondence.
- Infinite and Well-ordered Sets.
- Cardinals and Ordinals.
- Proved $\mathrm{card}(\mathbb{Q}) = \mathrm{card}(\mathbb{N})$.
- Proved $\mathrm{card}(\mathbb{R}) > \mathrm{card}(\mathbb{N})$.
- Hierarchy of Infinities.

Equality of Set Size: 1-1 Correspondence

How do we show that two sets are the same size?

For finite sets, this is straightforward counting.



For infinite sets, we must find a 1-1 correspondence.

Infinite Sets

We take the natural numbers and the even numbers

$$\mathbb{N} := \{1, 2, 3, ...\}$$

$$2\mathbb{N} := \{2, 4, 6, ...\}$$

By associating each number with its double,

$$n \in \mathbb{N} \longleftrightarrow 2n \in 2\mathbb{N}$$

we have a perfect 1-to-1 correspondence.

By Cantor's argument, the two sets are the same size:

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[2\mathbb{N}] .$$

Counterintuitive

But

$$\mathrm{card}[\mathbb{N}] = \mathrm{card}[2\mathbb{N}] \,.$$

is paradoxical: The set of natural numbers properly contains all the even numbers

$$2\mathbb{N} \subsetneqq \mathbb{N} \,.$$

But $\mathbb{N}$ also contains all the odd numbers:

$$\mathbb{N} = 2\mathbb{N} \uplus (2\mathbb{N} - 1) \,.$$

In an intuitive sense, $\mathbb{N}$ is larger than $2\mathbb{N}$.

Background

Cardinality is a *blunt instrument*:

The natural numbers, rationals and algebraic numbers all have the same cardinality.

So, $\aleph_0$ fails to discriminate between them.

Our aim is to define a number $m(A)$ for subsets A of $\mathbb{N}$ that corresponds to our intuition about the size or magnitude of A.

We define $m(A)$ as a surreal number.

Outline

Ordinal Numbers

Ordinal Numbers are used to describe the order type of well-ordered sets.

An ordinal may be defined as the set of ordinals that precede it. Thus 27 is the set $\{0, 1, 2, \dots, 26\}$.

The smallest infinite ordinal is ω, the order type of the set of natural numbers $\mathbb{N}$.

Indeed, ω can be identified with the set $\mathbb{N}$.

Von Neumann's Definition

Each ordinal number is the well-ordered set of all smaller ordinal numbers.

First few von Neumann ordinals

0 = { }		= ∅
1 = { 0 }		= {∅}
2 = { 0, 1 }		= { ∅, {∅} }
3 = { 0, 1, 2 }		= { ∅, {∅} , {∅, {∅}} }
4 = { 0, 1, 2, 3 }		= { ∅, {∅}, {∅, {∅}}, {∅, {∅}, {∅, {∅}}} }

For von Neumann, the successor of α is $\alpha \cup \{\alpha\}$.

A World of Ordinals from Empty Bags

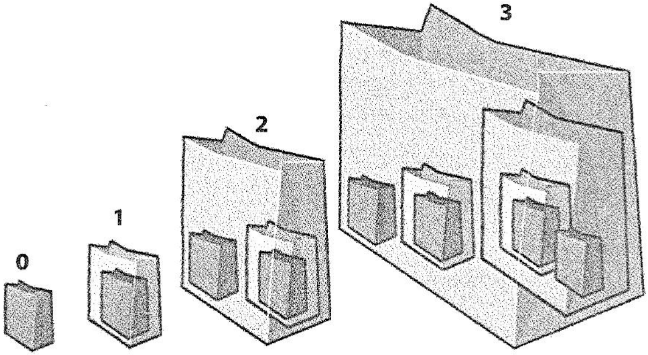


FIGURE 2.3 The empty set has no member, like an empty paper bag. But by putting the empty paper bag in a larger paper bag you can form big and bigger sets – the basis of our definition of number.

[Image: Source unknown].

The (proper) Class of Ordinal Numbers

Every well-ordered set has an ordinal number.

The class On of ordinal numbers is not a set.

If it were a set, it would be a member of itself, contradicting the strict ordering by membership.

Bertrand Russell noticed the contradiction. In 1903 he discussed it in his *Principles of Mathematics.*

Arithmetic on the Ordinals

The ordinals are non-commutative:

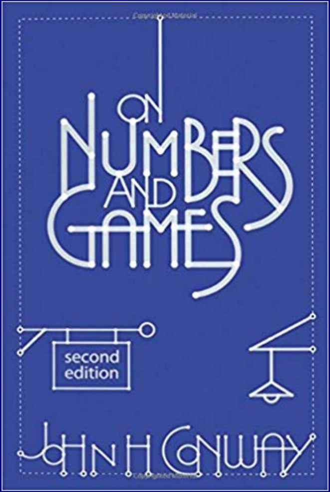
$$1 + \omega \neq \omega + 1$$

$$2\omega \neq \omega 2$$

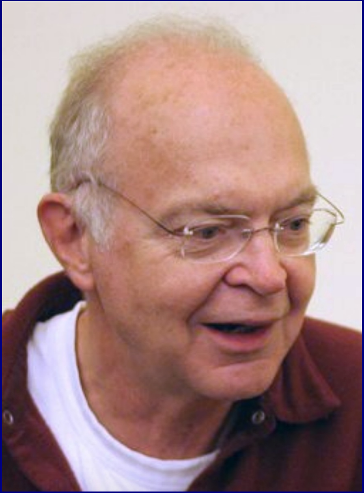
This is a poor basis for a calculus of transfinites.

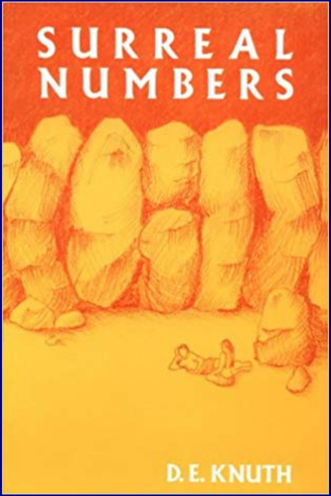
Outline

John H. Conway's ONAG [1976 / 2001]

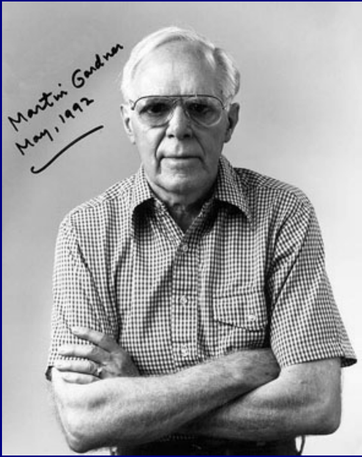


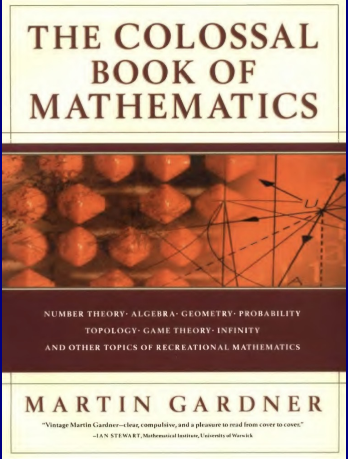
Donald Knuth's *Surreal Numbers* [1974]





Martin Gardner and *Surreal Numbers*





MG: "… the best friend mathematics ever had" [Colm Mulcahy]

Richard Dedekind (1831–1916)

Richard Dedekind defined irrational numbers by means of **cuts** of the rational numbers $\mathbb{Q}$.

For example, $\sqrt{2}$ is defined as (L, R), where

$$\begin{aligned} L &= \{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\} \\ R &= \{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\} \,. \end{aligned}$$

Irrational Numbers

For each irrational number there is a corresponding cut (L, R).

We can regard the cut as equivalent to the number.

There are rules to manipulate cuts that are equivalent to the arithmetical rules for numbers.

Irrational Numbers

For each irrational number there is a corresponding cut (L, R).

We can regard the cut as equivalent to the number.

There are rules to manipulate cuts that are equivalent to the arithmetical rules for numbers.

The surreal numbers are based upon a dramatic generalization of Dedekind's cuts.

Constructing the Surreals

The Surreal numbers No are constructed **inductively.**

- Every number x is defined by a pair of sets, the left set and the right set:

$$x = \{ L \mid R \}$$

- No element of L is greater than or equal to any element of R.

x is the **simplest** number between L and R.

Constructing the Surreals

In the beginning, we have no numbers, so L and R must both be void.

We start by defining 0 **as**

$$0 = \{\varnothing \mid \varnothing\} = \{\ \mid\ \}$$

Then 1, 2, 3 **and so on are defined as**

$$\{\,0 \mid\ \} = 1 \qquad \{\,1 \mid\ \} = 2 \qquad \{\,2 \mid\ \} = 3 \qquad \ldots$$

Negative numbers are defined inductively as

$$-x = \{-R \mid -L\,\}\,.$$

Constructing the Surreals

Dyadic fractions (of the form $m/2^n$) appear as

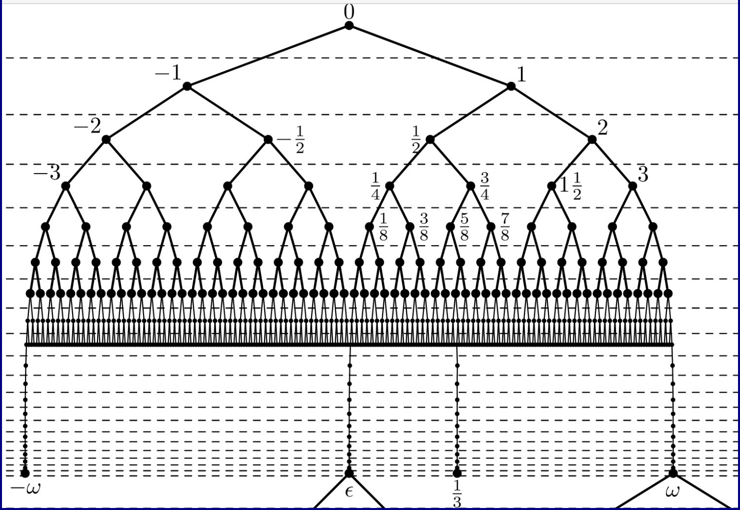
$$\{\,0\mid 1\,\} = \tfrac{1}{2} \quad \{\,1\mid 2\,\} = \tfrac{3}{2} \quad \{\,0\mid \tfrac{1}{2}\,\} = \tfrac{1}{4} \quad \{\,\tfrac{1}{2}\mid 1\,\} = \tfrac{3}{4} \quad \ldots$$

Over an infinite number of stages,
all the dyadic fractions emerge.

At that stage, all other real numbers appear.

Infinite and infinitesimal numbers also appear.

Surreal Numbers



Surreal network from 0 to the first infinite number ω.

[Image: Wikimedia Commons]

The First Infinite Number

The first infinite number ω is defined as

$$\omega = \{0, 1, 2, 3, \ldots \mid \ \}$$

We can also introduce

$$\begin{aligned}
\omega + 1 &= \{0, 1, 2, \ldots \ \omega | \ \} \\
\omega - 1 &= \{0, 1, 2, \ldots \ | \omega\} \\
2\omega &= \{0, 1, 2, \ldots \ \omega, \omega + 1, \ldots | \ \} \\
\tfrac{1}{2}\omega &= \{0, 1, 2, \ldots \ | \omega, \omega - 1, \ldots\} \,.
\end{aligned}$$

and many other more exotic numbers.

Manipulating Infinite Numbers

The surreal numbers behave beautifully:
The class No is a totally ordered Field.

We can define quantities like

$$\omega^2 \qquad \omega^\omega \qquad \sqrt{\omega} \qquad \log \omega$$

and many even stranger numbers.

Closing Lines of Knuth's Book

B. Alice! Feast your eyes on this!

$$\sqrt{\omega} \equiv \left(\{1, 2, 3, 4, \dots\}, \left\{ \frac{\omega}{1}, \frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega}{4}, \dots \right\} \right);$$

$$\sqrt{\epsilon} \equiv \left(\{\epsilon, 2\epsilon, 3\epsilon, 4\epsilon, \dots\}, \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \right).$$

A. (falling into his arms) Bill! Every discovery leads to more, and more!

B. (glancing at the sunset) There are infinitely many things yet to do ... and only a finite amount of time ... !

The Omnific Integers

Conway (ONAG, Ch. 5) defines the class Oz of **omnific integers:** $x \in$ No is an omnific integer if

$$x = \{x - 1 | x + 1\} .$$

- $\mathbb{Z} \subset$ Oz and On $\subset$ Oz.
- No is the fraction field of Oz.
- There are no infinitesimals in Oz.
- Every surreal number is distant at most 1 from an omnific integer.

Omnifics are the appropriate integers for No.

The Surnatural Numbers

We assume that the magnum function maps sets of natural numbers into the positive omnific numbers

$$m : \mathscr{P}(\mathbb{N}) \rightarrow \textbf{Nn} := \textbf{Oz}^+ \, .$$

Nn is the set of surnatural numbers.

All numbers of the form $r \cdot \omega^\beta$ are in **Nn**.

Thus, ω is an even number, since $\omega/2 \in$ **Nn**; a multiple of 3, since $\omega/3 \in$ **Nn**; and so on.

Moreover, $\sqrt[k]{\omega} \in$ **Nn**, so ω is a perfect square, a perfect cube, and so on.

Desiderata for the Magnum Function

- **For a finite subset A we have $m(A) = \mathrm{card}(A)$**
- **For a proper subset A of B we have**

$$A \subsetneqq B \implies m(A) < m(B)\,.$$

- **For the odd and even natural numbers**

$$\begin{aligned} 2\mathbb{N}-1 &= \{1,3,5,\dots\} & \qquad m(2\mathbb{N}-1) &= \tfrac{1}{2}m(\mathbb{N}) \\ 2\mathbb{N} &= \{2,4,6,\dots\} & \qquad m(2\mathbb{N}) &= \tfrac{1}{2}m(\mathbb{N}) \end{aligned}$$

- **Ten desiderata listed in L& M.**

The Magnum Form

For any subset A of natural numbers, we seek two sets, L_A and R_A such that

$$m(A) = \{ \ L_A \mid R_A \ \}$$

Clearly, this should hold if

- **The sets in L_A are all the subsets of A ;**
- **The sets in R_A are all the supersets of A .**

Outline

Isobaric Equivalence

A **fenestration** $\mathcal{W} = \{W_k : k \in \mathbb{N}\}$ is a collection of disjoint finite sets or 'windows', all of length L, with

$$\mathbb{N} = \biguplus_{k\in\mathbb{N}} W_k \,, \qquad \text{where} \qquad W_k = \{(k-1)L+1, \ldots, kL\}\,.$$

The **weights** of A are $\mathbf{w}_A = \langle \#(A \cap W_1), \#(A \cap W_2), \ldots \rangle$.

Two sets A_1, A_2 are **isobaric** if, for some $L \in \mathbb{N}$, they have equal weight sequences, i.e., if $\mathbf{w}_{A_1} = \mathbf{w}_{A_2}$.

Isobary is an equivalence relation, denoted $A_1 \cong A_2$.

Beta-algebras over $\mathbb{N}$

A beta-algebra over the natural numbers is
a family of subsets of $\mathbb{N}$ that is closed under finite unions, complements and isobaric equivalence.

Definition

A family $\mathscr{B}$ of subsets of $\mathbb{N}$ is a β-algebra if

1. The union of any pair of sets A_1 and A_2 in $\mathscr{B}$ is in $\mathscr{B}$,
2. The complement of any set A in $\mathscr{B}$ is in $\mathscr{B}$,
3. If $A_1 \in \mathscr{B}$ and $A_2 \cong A_1$, then $A_2 \in \mathscr{B}$.

Magnum Spaces

A **magnum space** over $\mathbb{N}$ is a triplet $(\mathbb{N}, \mathscr{B}, m)$ consisting of the set $\mathbb{N}$, a β-algebra $\mathscr{B}$ of subsets of $\mathbb{N}$ and a function $m : \mathscr{B} \to$ Nn, the *magnum*, such that,

1. $m(\varnothing) = 0$ and $m(\mathbb{N}) = \omega$.
2. $m(\{x\}) = 1$ for all singletons $\{x\} \in \mathscr{B}$.
3. For all $A_1, A_2 \in \mathscr{B}$, if $A_1 \cong A_2$ then $m(A_1) = m(A_2)$.
4. $m(A_1 \uplus A_2) = m(A_1) + m(A_2)$ for disjoint sets in $\mathscr{B}$.

Euclidean Principle

The Euclidean Principle holds for all magnum spaces:

Theorem

If A_1, A_2 are in $\mathscr{B}$ with $A_1 \subset A_2$, then $m(A_1) < m(A_2)$.

This is a fundamental requirement for magnums.
The proof follows easily from the definitions.

We will define the magnum to ensure that it is true.

Outline

The Magnum Form

Given a set A we seek two sets L_A and R_A such that

$$m(A) = \{L_A \mid R_A\}$$

is the magnum of A.

The challenge is to construct m so that $(\mathbb{N}, \mathscr{P}(\mathbb{N}), m)$ is a magnum space.

We then know that the Euclidean Principle holds.

We seek a general expression in the form

$$m(A) = \{m(B) : B \subset A \mid m(C) : A \subset C\} ,$$

This guarantees the Euclidean Principle.

However, this requires knowledge
of the magnums of B and C.

We must to construct $m(A)$ in incremental fashion.

We use the magnums of ‘old’ sets to generate the magnums of ‘new’ sets.

For each ordinal number α, we define three families:

- $\mathscr{M}_\alpha$: sets **magnumbered** on or before day α,
- $\mathscr{N}_\alpha$: sets **magnumbered** on day α, and
- $\mathscr{O}_\alpha$: sets **magnumbered** before day α.

The last two families combine to give the first:

$$\mathscr{M}_\alpha = \mathscr{N}_\alpha \uplus \mathscr{O}_\alpha = \mathscr{O}_{\alpha+1} \, .$$

For each ordinal γ, we define a premagnum,

$$m_\gamma(A) = \{m(B) : B \in \mathscr{O}_\gamma, B \subset A \mid m(C) : C \in \mathscr{O}_\gamma, A \subset C\}\,.$$

The proper subsets B and supersets C range over all sets "magnumbered" prior to day γ.

When a stage $\gamma = \alpha$ is reached where $m_\gamma(A)$ cannot undergo further changes, we define

$$m(A) := m_\alpha(A)$$

and call α the birthday of $m(A)$.

Outline

When is the magnum of a subset of $\mathbb{N}$ first defined?
To answer , we consider the ordinals as they arise.

Day 0: The magnum of $\varnothing$ is defined to be 0.

Day 1: Magnums of all singletons $\{n\}$ defined to be 1.

Day 2: Magnums of all doubletons $\{m, n\}$ equal to 2.

Day n: The magnums of all sets with n elements are defined to be n.

All finite subsets of $\mathbb{N}$ are defined on finite days.
Their magnums are all the finite ordinal numbers.

Day ω: The set $\mathbb{N}$ is given a magnum on this day: $m(\mathbb{N}) = \omega$, the first infinite magnum.

$\omega + 1$: All "co-singletons", sets of the form $\mathbb{N} \setminus \{k\}$, are assigned the value $\omega - 1$.

$\omega + n$: All sets with complements having magnum n are assigned the value $\omega - n$.

Before day 2ω, all finite and cofinite sets, together with $\mathbb{N}$, have magnums assigned.

Day 2ω: The new magnum is $\omega/2$. The obvious candidate for this magnum is $2\mathbb{N}$ so $m(2\mathbb{N}) = \omega/2$.

- All sets isobaric to $2\mathbb{N}$ have magnum $\omega/2$. This is an uncountable collection of sets.

Calendar for Magnumbering Sets

Day #	Magnum	Sets
0	0	$\varnothing$
1	1	$\{k\}$, $\cong$
2	2	$\{k, \ell\}$, $\cong$
3	3	$\{k, \ell, m\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
n	n	$\{m_1, m_2, \ldots, m_n\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
ω	ω	$\mathbb{N}$
$\omega + 1$	$\omega - 1$	$\mathbb{N} \setminus \{k\}$, $\cong$
$\omega + 2$	$\omega - 2$	$\mathbb{N} \setminus \{k, \ell\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
$\omega + n$	$\omega - n$	$\mathbb{N} \setminus \{m_1, m_2, \ldots, m_n\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
2ω	$\frac{\omega}{2}$	$2\mathbb{N}$, $2\mathbb{N} - 1$, $\cong$
$2\omega + 1$	$\frac{\omega}{2} + 1$ $\frac{\omega}{2} - 1$	$2\mathbb{N} \uplus \{k\}$, $\cong$ $2\mathbb{N} \setminus \{k\}$, $\cong$
$2\omega + 2$	$\frac{\omega}{2} + 2$ $\frac{\omega}{2} - 2$	$2\mathbb{N} \uplus \{k, \ell\}$, $\cong$ $2\mathbb{N} \setminus \{k, \ell\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
3ω	$\frac{\omega}{4}$ $\frac{3}{4}\omega$	$4\mathbb{N}$, $\cong$ $\mathbb{N} \setminus 4\mathbb{N}$, $\cong$
$3\omega + 1$	$\frac{\omega}{4} + 1$ $\frac{\omega}{4} - 1$ $\frac{3}{4}\omega + 1$ $\frac{3}{4}\omega - 1$	$4\mathbb{N} \uplus \{k\}$, $\cong$ $4\mathbb{N} \setminus \{k\}$, $\cong$ $(\mathbb{N} \setminus 4\mathbb{N}) \uplus \{k\}$, $\cong$ $(\mathbb{N} \setminus 4\mathbb{N}) \setminus \{k\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
4ω	$\frac{1}{8}\omega$ $\frac{3}{8}\omega$ $\frac{5}{8}\omega$ $\frac{7}{8}\omega$	$8\mathbb{N}$, $\cong$ $8\mathbb{N} \uplus (8\mathbb{N} - 1) \uplus (8\mathbb{N} - 1)$ $\cong$ $\mathbb{N} \setminus [8\mathbb{N} \uplus (8\mathbb{N} - 1) \uplus (8\mathbb{N} - 1)]$, $\cong$ $\mathbb{N} \setminus 8\mathbb{N}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
ω^2	$\frac{j}{k}\omega$ $\cdots$ $\sqrt{\omega}$	$\biguplus_{m=0}^{j-1}(k\mathbb{N} - m)$, $\cong$ $\cdots$ $\mathbb{N}^2$

Calendar for Magnumbering Sets

Day #	Magnum	Sets
0	0	$\varnothing$
1	1	$\{k\}$, $\cong$
2	2	$\{k,\ell\}$, $\cong$
3	3	$\{k,\ell,m\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
n	n	$\{m_1,m_2,\ldots,m_n\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
ω	ω	$\mathbb{N}$
$\omega+1$	$\omega-1$	$\mathbb{N}\setminus\{k\}$, $\cong$
$\omega+2$	$\omega-2$	$\mathbb{N}\setminus\{k,\ell\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
$\omega+n$	$\omega-n$	$\mathbb{N}\setminus\{m_1,m_2,\ldots,m_n\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
2ω	$\frac{\omega}{2}$	$2\mathbb{N}$, $2\mathbb{N}-1$, $\cong$

2ω	$\frac{\omega}{2}$	$2\mathbb{N}$, $2\mathbb{N}-1$, $\cong$
$2\omega+1$	$\frac{\omega}{2}+1$ $\frac{\omega}{2}-1$	$2\mathbb{N}\uplus\{k\}$, $\cong$ $2\mathbb{N}\setminus\{k\}$, $\cong$
$2\omega+2$	$\frac{\omega}{2}+2$ $\frac{\omega}{2}-2$	$2\mathbb{N}\uplus\{k,\ell\}$, $\cong$ $2\mathbb{N}\setminus\{k,\ell\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
3ω	$\frac{\omega}{4}$ $\frac{3}{4}\omega$	$4\mathbb{N}$, $\cong$ $\mathbb{N}\setminus 4\mathbb{N}$, $\cong$
$3\omega+1$	$\frac{\omega}{4}+1$ $\frac{\omega}{4}-1$ $\frac{3}{4}\omega+1$ $\frac{3}{4}\omega-1$	$4\mathbb{N}\uplus\{k\}$, $\cong$ $4\mathbb{N}\setminus\{k\}$, $\cong$ $(\mathbb{N}\setminus 4\mathbb{N})\uplus\{k\}$, $\cong$ $(\mathbb{N}\setminus 4\mathbb{N})\setminus\{k\}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
4ω	$\frac{1}{8}\omega$ $\frac{3}{8}\omega$ $\frac{5}{8}\omega$ $\frac{7}{8}\omega$	$8\mathbb{N}$, $\cong$ $8\mathbb{N}\uplus(8\mathbb{N}-1)\uplus(8\mathbb{N}-1)$ $\cong$ $\mathbb{N}\setminus[8\mathbb{N}\uplus(8\mathbb{N}-1)\uplus(8\mathbb{N}-1)]$, $\cong$ $\mathbb{N}\setminus 8\mathbb{N}$, $\cong$
$\cdots$	$\cdots$	$\cdots$
ω^2	$\frac{j}{k}\omega$ $\cdots$ $\sqrt{\omega}$	$\biguplus_{m=0}^{j-1}(k\mathbb{N}-m)$, $\cong$ $\cdots$ $\mathbb{N}^2$

B. (glancing at the sunset) There are infinitely many things yet to do … and only a finite amount of time … **!**

Outline

Difficulties with Limits

In ONAG (page 43), Conway states that we cannot assume the limit of the sequence $(1, 2, 3, \dots)$ is ω.

Therefore, we cannot conclude that $m(\mathbb{N}) = \omega$.

However, we can consistently define $m(\mathbb{N}) = \omega$.

Extending Functions from $\mathbb{N}$ to Nn

We define the **counting function** $\kappa_A : \mathbb{N} \to \mathbb{N}$ to be

$$\kappa_A(n) = \text{Number of terms less than or equal to } n.$$

Sometimes, the extension to Nn is obvious:

$$\kappa : n \mapsto n^2 \,, \; n \in \mathbb{N} \qquad \text{to} \qquad \kappa : \nu \mapsto \nu^2 \,, \; \nu \in \textbf{Nn} \,.$$

so we have $\kappa(\omega) = \omega^2$ and so on.

The **Transfer Axiom** generalizes this idea.

The Transfer Axiom

Transfer Axiom (General Form).
Properties expressed by formulas or statements that hold for all real numbers can be *transferred* to hold also for surreal numbers.

For us, a restricted form of the axiom suffices:

Transfer Axiom (Special Form).
For all monotone functions $f : \mathbb{N} \to \mathbb{N}$, there is an extension $f :$ **Nn** $\to$ **Nn**, such that
$[\forall n \in \mathbb{N} : f_1(n) \leq f_2(n)] \implies [\forall \nu \in \mathbf{Nn} : f_1(\nu) \leq f_2(\nu)]$.

Defining the Magnum of A

The defining function is

$$\alpha_A(n) := a_n \,.$$

The counting function may be expressed as

$$\kappa_A(n) = \lfloor \alpha_A^{-1}(n) \rfloor \,.$$

If κ_A is extended to Nn, we can define the magnum of A to be:

$$m(A) := \kappa_A(\omega)$$

This definition is compatible with the iterative genetic assignment of magnums.

Theorems from Transfer Axiom

Finite Additivity Theorem:
Let $A_1, A_2, \ldots, A_n$ be mutually disjoint sets in $\mathfrak{M}$.
The magnum of the union is the sum of the magnums:

$$m\left(\biguplus_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} m(A_k)\,.$$

Window Theorem:
Let A_1 have magnum $m(A_1)$.
Then $[A_2 \cong A_1] \implies [m(A_1) = m(A_2)]$.

Magnum Space Theorem:
The Triplet $(\mathbb{N}, \mathfrak{M}, m)$ is a magnum space.

Outline

Examples of magnums

$$A = k\mathbb{N} = \{kn : n \in \mathbb{N}\} \qquad m(A) = \omega/k$$

$$A = \mathbb{N}^{(2)} = \{n^2 : n \in \mathbb{N}\} \qquad m(A) = \sqrt{\omega}$$

$$A = \mathbb{N}^{(k)} = \{n^k : n \in \mathbb{N}\} \qquad m(A) = \sqrt[k]{\omega}$$

$$A = \{k^n : n \in \mathbb{N}\} \qquad m(A) = \lfloor \log_k \omega \rfloor \,.$$

The general arithmetic sequence $A = \{\alpha(n) : n \in \mathbb{N}\}$ has $\alpha(n) = kn + \ell$ and $\alpha^{-1}(n) = (n - \ell)/k$ so

$$m(A) = \kappa_A(\omega) = \left\lfloor \frac{\omega}{k} - \frac{\ell}{k} \right\rfloor .$$

The set $A = \mathbb{N}^{(2)} \cup \mathbb{N}^{(3)}$, containing all squares and cubes, has magnum

$$m(\mathbb{N}^{(2)} \cup \mathbb{N}^{(3)}) = \left(\sqrt[2]{\omega} + \sqrt[3]{\omega} - \sqrt[6]{\omega} \right) .$$

Conclusion

There are many sets for which we are unable to calculate the magnums; for example, the set Od_2, whose elements are all numbers having an odd number of binary digits.

The natural density of this set oscillates between values that asymptote to $\frac{1}{3}$ and $\frac{2}{3}$, never tending to a limit. Another axiom may be required to determine $m(\mathrm{Od}_2)$ uniquely.

The theory developed here is amenable to extension beyond subsets of $\mathbb{N}$. For example, it is reasonable to expect that $m(\mathbb{N} \times \{1, 2\}) = 2\omega$ and $m(\mathbb{N} \times \mathbb{N}) = \omega^2$.

Outline

Opportunities

- Many open challenges in analysis over surreals.
- Surreals may be of value in physics.
- Great projects for students.

Reference

Peter Lynch & Michael Mackey, 2023:
Counting Sets with Surreals.
https://arxiv.org/abs/2311.09951

Thank you