Laplace Transform Integration and the Slow Equations

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Outline

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The Laplace Transform: Definition

For a function of time $f(t),\ t \geq 0$, the LT is defined as

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,.$$

Here, s is complex and $\hat{f}(s)$ is a complex function of s.

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- The domain of the function $f(t)$ is $\mathcal{D} = [0, +\infty)$.
- The kernel of the transform is $K(s, t) = \exp(-st)$.
- The domain of the LT $\hat{f}(s)$ is the complex s-plane.

Recovering the Original Function

For the LT, the inversion formula is

$$f(t) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} e^{st}\,\hat{f}(s)\,\mathrm{d}s\,.$$

where $\mathcal{C}_1$ is a contour in the s-plane:

- $\mathcal{C}_1$ is parallel to the imaginary axis.
- $\mathcal{C}_1$ is to the right of all singularities of $\hat{f}(s)$.

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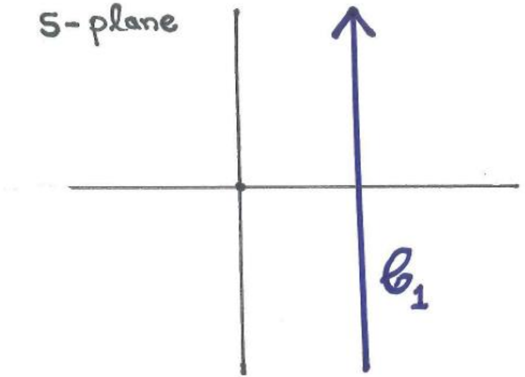
- $\mathcal{C}_1$ is to the right of all singularities of $\hat{f}(s)$.

For the functions that we consider, the singularities are poles on the imaginary axis.

Thus, the contour $\mathcal{C}_1$ must be in the right half-plane.

Contour for inversion of Laplace Transform



A Simple Oscillation

Let $f(t)$ have a single harmonic component

$$f(t) = \alpha \, exp(i\omega t)$$

The LT of $f(t)$ has a simple pole at $s = i\omega$:

$$\hat{f}(s) = \frac{\alpha}{s - i\omega} ,$$

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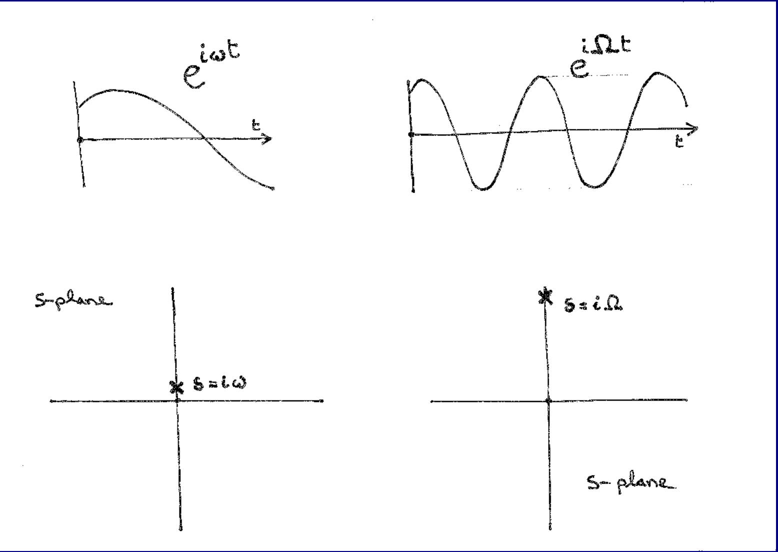
A pure oscillation in time transforms to a **holomorphic function**, with a single pole.

The frequency of the oscillation determines the position of the pole.



LF and HF oscillations and their transforms

The inverse transform of $\hat{f}(s)$ is

$$f(t) = \frac{1}{2\pi i} \int_{\mathcal{C}_1} e^{st}\, \hat{f}(s)\, \mathrm{d}s = \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{\alpha \exp(st)}{s - i\omega}\, \mathrm{d}s\,.$$

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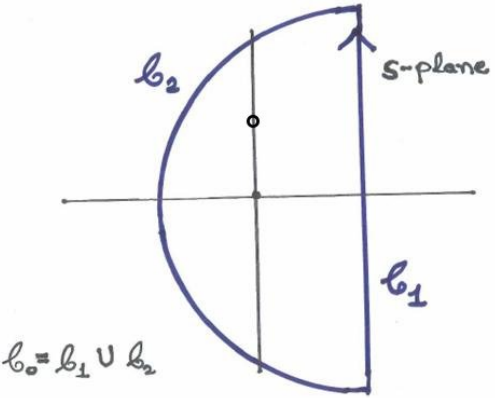
We augment $\mathcal{C}_1$ by a semi-circular arc $\mathcal{C}_2$ in the left half-plane. Denote the resulting closed contour by

$$\mathcal{C}_0 = \mathcal{C}_1 \cup \mathcal{C}_2\,.$$

In cases of interest, we can show that this leaves the value of the integral unchanged (see Doetsch, 1977).

Then $f(t)$ is an integral around a closed contour $\mathcal{C}_0$.

Closed Contour



Contribution from $\mathcal{C}_2$ vanishes in limit of infinite radius

For an integral around a closed contour,

$$f(t) = \frac{1}{2\pi i} \oint_{\mathcal{C}_0} \frac{\alpha \exp(st)}{s - i\omega} \, \mathrm{d}s \,,$$

we can apply the **residue theorem**:

For an integral around a closed contour,

$$f(t) = \frac{1}{2\pi i}\oint_{\mathcal{C}_0} \frac{\alpha\,\exp(st)}{s - i\omega}\,\mathrm{d}s\,,$$

we can apply the residue theorem:

$$f(t) = \sum_{\mathcal{C}_0}\left[\text{Residues of }\left(\frac{\alpha\,\exp(st)}{s - i\omega}\right)\right]$$

so $f(t)$ is the sum of the residues of the integrand within the contour $\mathcal{C}_0$.

There is just one pole, at $s = i\omega$. The residue is

$$\lim_{s \to i\omega} (s - i\omega) \left(\frac{\alpha \exp(st)}{s - i\omega} \right) = \alpha \exp(i\omega t)$$

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So we recover the input function:

$$f(t) = \alpha\,\exp(i\omega t)$$

A Two-Component Oscillation

Let $f(t)$ have two harmonic components

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The LT is a linear operator, so the transform of $f(t)$ is

$$\hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}\,,$$

which has two simple poles, at $s = i\omega$ and $s = i\Omega$.

- **The LF pole, at $s = i\omega$, is close to the origin.**
- **The HF pole, at $s = i\Omega$, is far from the origin.**

Again

$$\hat{f}(s) = \frac{a}{s - i\omega} + \frac{A}{s - i\Omega}\,.$$

The inverse transform of $\hat{f}(s)$ is

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\oint_{\mathcal{C}_0} \frac{a\exp(st)}{s - i\omega}\,\mathrm{d}s + \frac{1}{2\pi i}\oint_{\mathcal{C}_0} \frac{A\exp(st)}{s - i\Omega}\,\mathrm{d}s \\ &= a\exp(i\omega t) + A\exp(i\Omega t)\,. \end{aligned}$$

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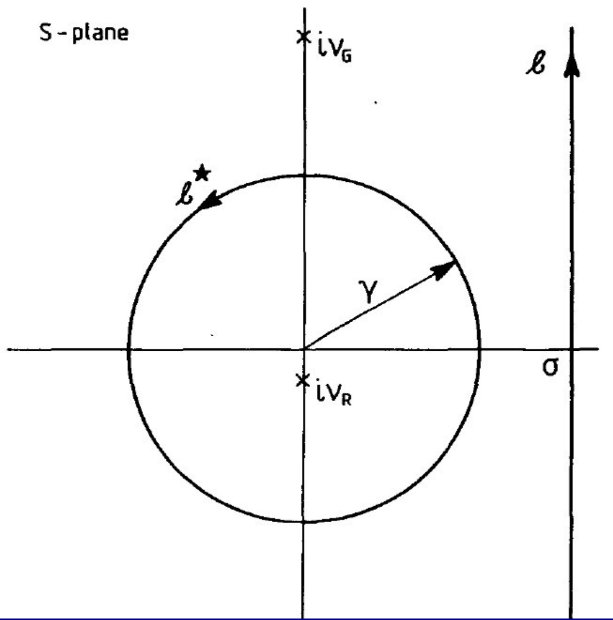
We now replace $\mathcal{C}_0$ by a circular contour $\mathcal{C}^\star$ centred at the origin, with radius γ such that $|\omega| < \gamma < |\Omega|$.


S - plane
iV_G
ℓ
ℓ^*
γ
σ
iV_R

We denote the modified operator by $\mathcal{L}^\star$.

Since the pole $s = i\omega$ falls within the contour $\mathcal{C}^\star$, it contributes to the integral.

Since the pole $s = i\Omega$ falls outside the contour $\mathcal{C}^\star$, it makes *no contribution*.

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Therefore,

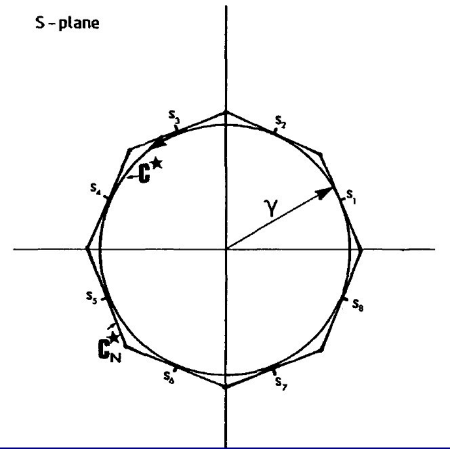
$$f^\star(t) \equiv \mathcal{L}^\star\{\hat{f}(s)\} = \frac{1}{2\pi i} \oint_{\mathcal{C}^\star} \frac{a \exp(st)}{s - i\omega}\, \mathrm{d}s = a \exp(i\omega t)\,.$$

We have filtered $f(t)$: the function $f^\star(t)$ is the LF component of $f(t)$. The HF component is gone.

Outline

Approximating the Contour $\mathcal{C}^\star$

We replace the circle $\mathcal{C}^\star$ by an N-gon $\mathcal{C}^\star_N$:



Numerical approximation: the inverse

$$\mathcal{L}^{\star}\{\hat{f}(s)\} = \frac{1}{2\pi i}\oint_{\mathcal{C}^{\star}} \exp(st)\,\hat{f}(s)\,\mathrm{d}s$$

is approximated by the summation

$$\mathcal{L}^{\star}_{N}\{\hat{f}(s)\} = \frac{1}{2\pi i}\sum_{n=1}^{N} \exp(s_n t)\,\hat{f}(s_n)\,\Delta s_n$$

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We introduce a correction factor, and arrive at:

$$\mathcal{L}^{\star}_{N}\{\hat{f}(s)\} = \frac{1}{N}\sum_{n=1}^{N} \exp_N(s_n t)\,\hat{f}(s_n)\,s_n$$

Here $\exp_N(z)$ is the N-term Taylor expansion of $\exp(z)$.

(For details, see Clancy and Lynch, 2011a)

Outline

Applying LT to an ODE

We consider a nonlinear ordinary differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} + i\omega w + n(w) = 0 \qquad w(0) = w_0$$

The LT of the equation is

$$(s\hat{w} - w_0) + i\omega\hat{w} + \frac{n_0}{s} = 0\,.$$

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We can immediately solve for the transform solution:

$$\hat{w}(s) = \frac{1}{s+i\omega}\left[w_0 - \frac{n_0}{s}\right] = \left(\frac{w_0}{s+i\omega}\right) + \frac{n_0}{i\omega}\left(\frac{1}{s+i\omega} - \frac{1}{s}\right)$$

There are two poles, at $s = -i\omega$ and at $s = 0$.

The solution is:

$$w^\star(t) = \begin{cases} \left(w_0 + \dfrac{n_0}{i\omega}\right)\exp(-i\omega t) - \dfrac{n_0}{i\omega} & : \quad |\omega| < \gamma \\ -\dfrac{n_0}{i\omega} & : \quad |\omega| > \gamma \end{cases}$$

High frequencies are filtered out.

This corresponds to dropping the time derivative and holding the nonlinear term at its initial value: the criterion for nonlinear normal mode initialization.

Outline

A General NWP Equation

We write the general NWP equations symbolically as

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0}$$

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We apply the Laplace transform to get

$$(s\,\hat{\mathbf{X}} - \mathbf{X}_0) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_0 = \mathbf{0}$$

where $\mathbf{X}_0$ is the initial value of $\mathbf{X}$ and $\mathbf{N}_0 = \mathbf{N}(\mathbf{X}_0)$ is held constant at its initial value.

Now we take $n\Delta t$ to be the initial time:

$$(s\hat{\mathbf{X}} - \mathbf{X}^n) + i\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}^n = \mathbf{0}$$

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Laplace transform integration of the shallow water equations. Part 1: Eulerian formulation and Kelvin waves

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Laplace transform integration of the shallow water equations. Part 2: Lagrangian formulation and orographic resonance

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Phase Errors of SI and LT Schemes

Consider the phase error of the oscillation equation

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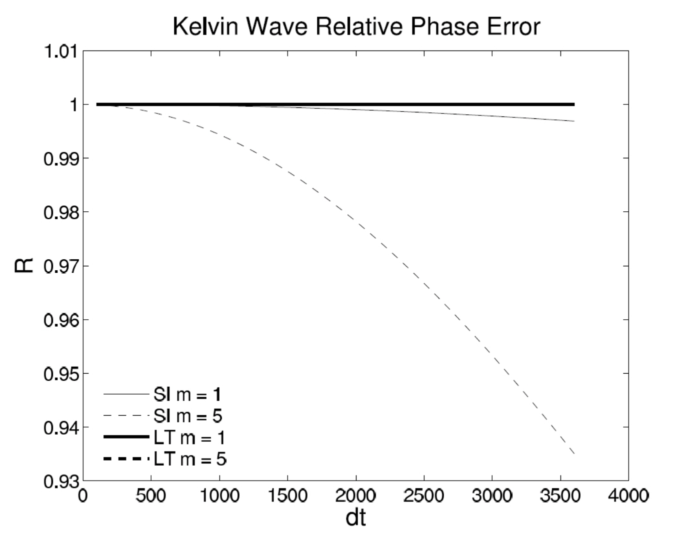
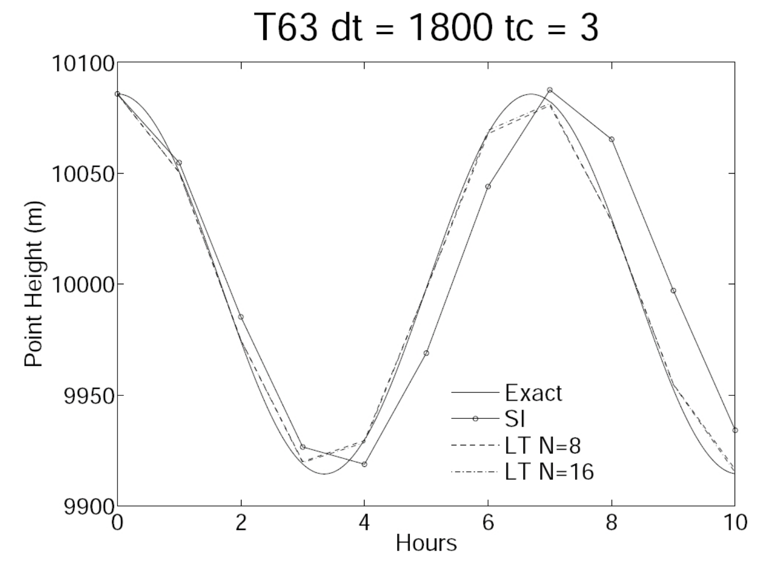
For the LT scheme, the corresponding error is

$$R_{\mathrm{LT}} = 1 - \frac{1}{N!}\,(\omega\Delta t)^N$$

Even for modest values of N, this is negligible.



Relative phase errors for semi-implicit (SI) and Laplace transform (LT) schemes for Kelvin waves $m = 1$ and $m = 5$.



Hourly height at $0.0°$E, $0.9°$N over 10 hours, with $\tau_c = 3\,$h.

Outline

- Basic Theory
- Numerical Inversion
- Ordinary Differential Equations
- Application to Numerical Weather Prediction
- Kelvin Waves & Phase Errors
- Lagrangian Formulation
- Orographic Resonance
- Analytical Inversion
- Results

Lagrangian Formulation

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The general form of the equation is

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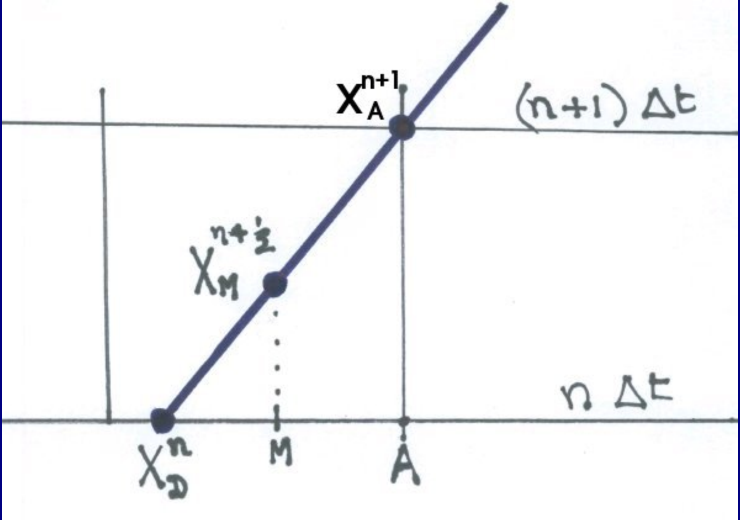
We *re-define* the Laplace transform to be the integral in time *along the trajectory of a fluid parcel*:

$$\hat{\mathbf{X}}(s) \equiv \int_{\mathcal{T}} e^{-st}\,\mathbf{X}(t)\,\mathrm{d}t$$



We compute $\mathcal{L}$ along a fluid trajectory $\mathcal{T}$.

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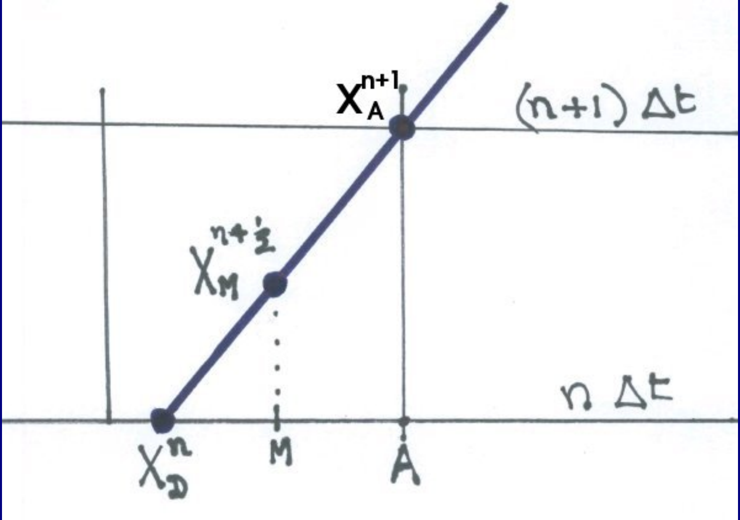
The equations thus transform to

$$(s\hat{\mathbf{X}} - \mathbf{X}_{\mathrm{D}}^{n}) + i\,\mathbf{L}\hat{\mathbf{X}} + \frac{1}{s}\mathbf{N}_{\mathrm{M}}^{n+\frac{1}{2}} = \mathbf{0}$$

where we evaluate nonlinear terms at a **mid-point**, interpolated in space and extrapolated in time.



Departure point, arrival point and mid-point.

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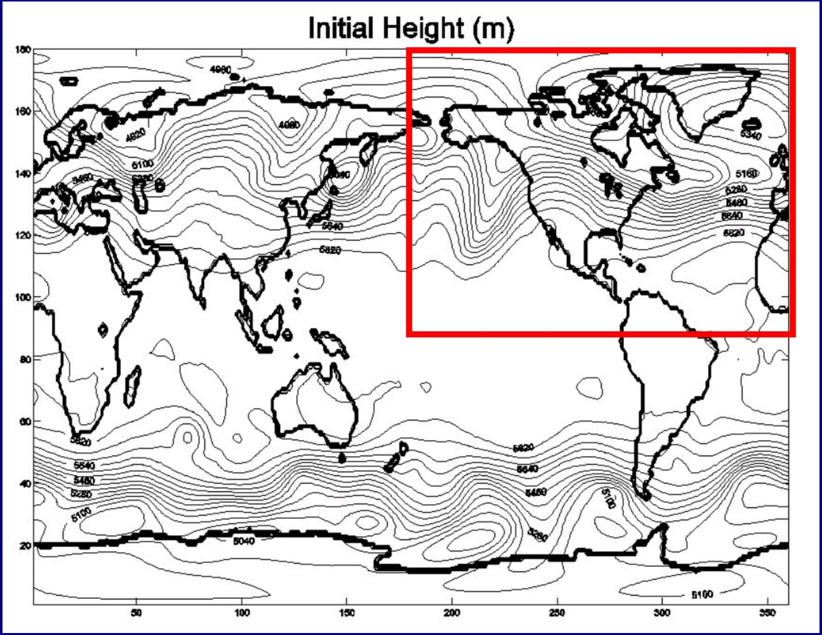
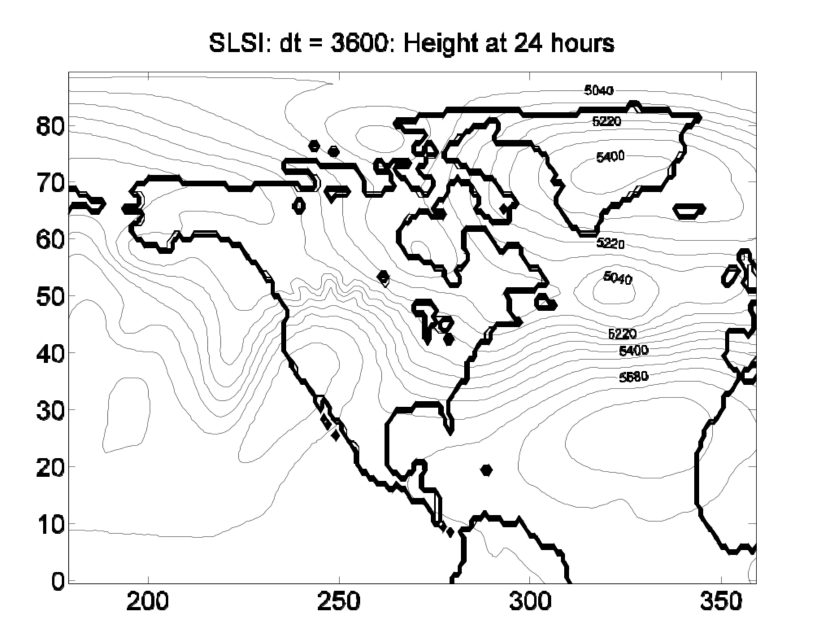
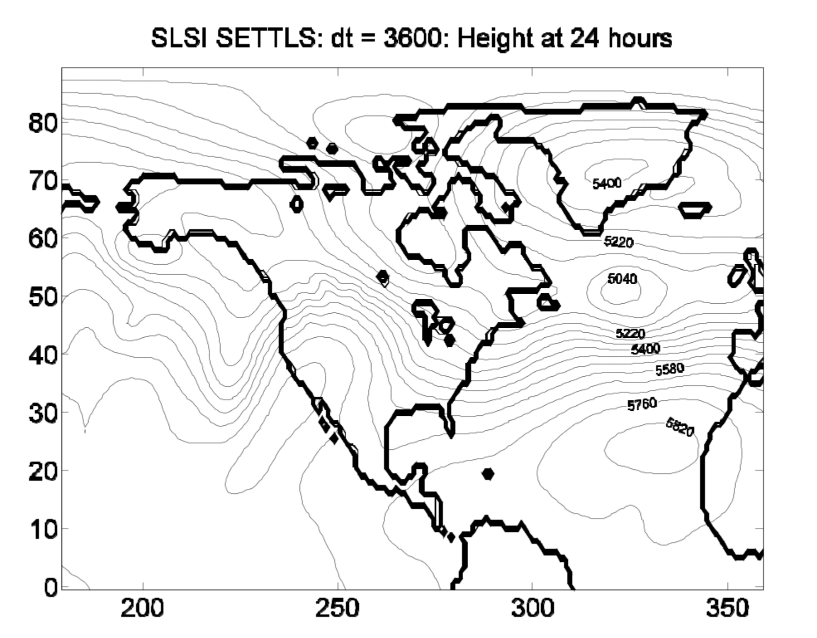
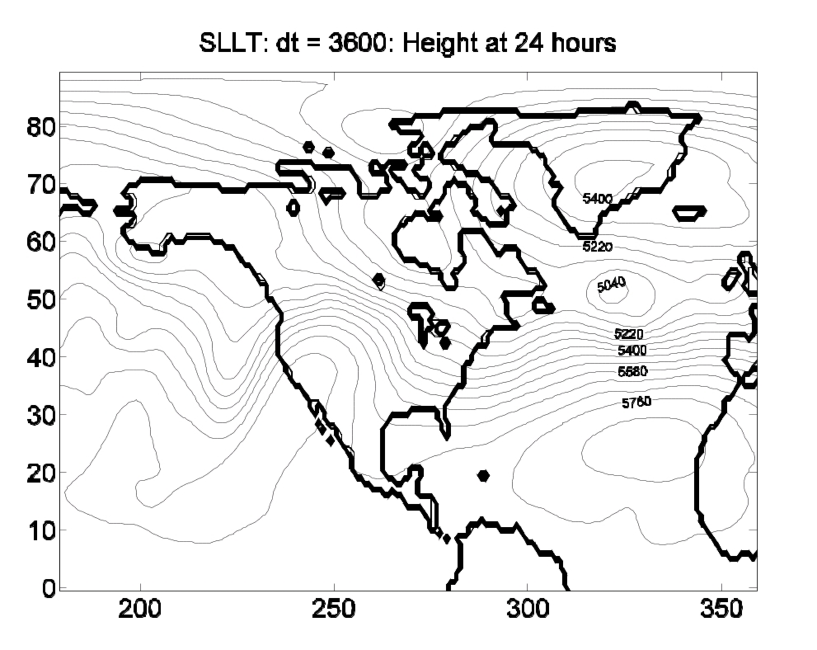
- Running at T119 resolution
- Shows LT method has benefits over SI scheme.

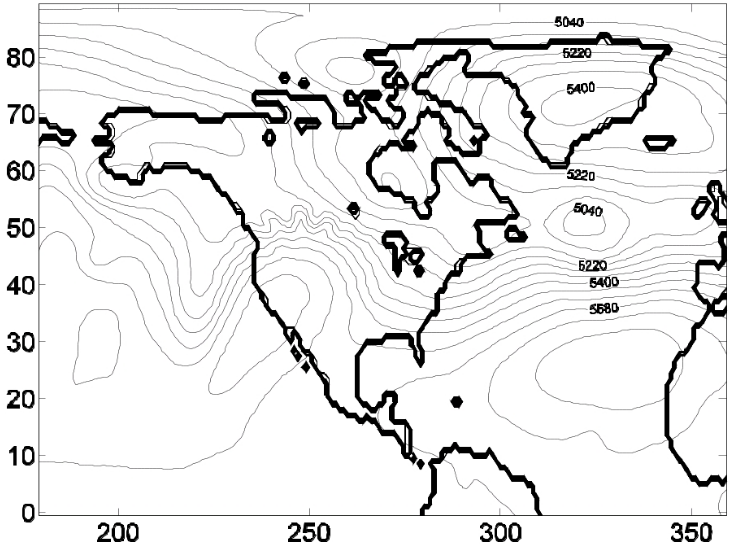

Initial Height (m)
4980
4920
5100
5460
5640
5820
5340
5160
5280
5040
50
100
150
200
250
300
350
180
160
140
120
100
80
60
40
20

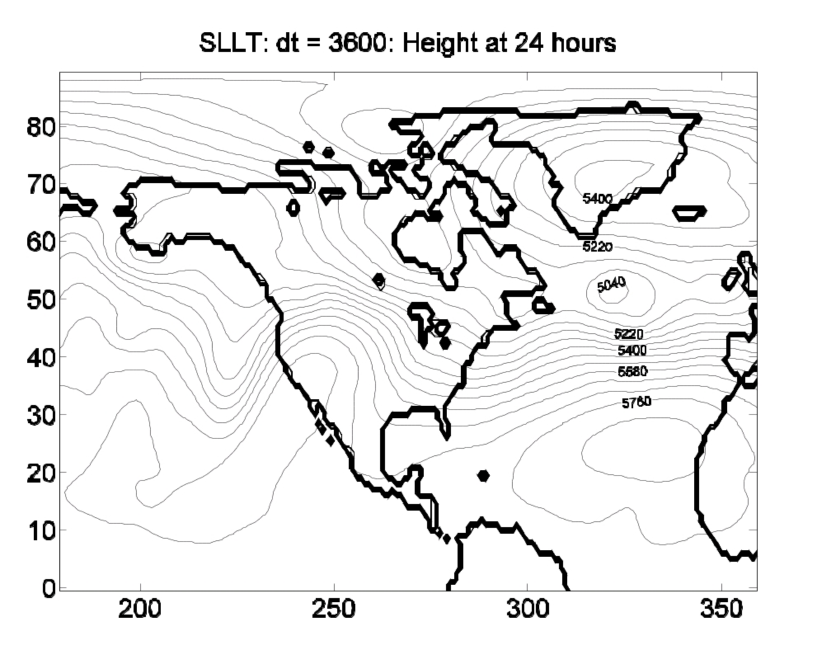

SLSI: dt = 3600: Height at 24 hours
5040
5220
5400
5220
5040
5220
5400
5580
80
70
60
50
40
30
20
10
0
200
250
300
350


SLSI SETTLS: dt = 3600: Height at 24 hours
80
70
60
50
40
30
20
10
0
200
250
300
350
5400
5220
5040
5220
5400
5580
5760
5820


SLLT: dt = 3600: Height at 24 hours
80
70
60
50
40
30
20
10
0
200
250
300
350
5400
5220
5040
5220
5400
5580
5760

SLSI: dt = 3600: Height at 24 hours





SLLT: dt = 3600: Height at 24 hours
80
70
60
50
40
30
20
10
0
200
250
300
350
5400
5220
5040
5220
5400
5580
5760

Outline

Analytical Inversion

We now consider the LT scheme with the inverse computed analytically.

This yields a filtered system. We relate it to the filtering schemes of Daley (1980).

The procedure requires explicit knowledge of the positions of the poles of the function to be inverted.

For the Eulerian model, this is simple.

For the Lagrangian model, a transformation to normal mode space is required.

We can write the general, diagonalized system:

$$\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}t} + i\,\Omega W + \mathbf{N}_{\mathrm{W}}(\mathbf{X}) = 0\,.$$

We separate this into LF and HF components:

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} + i\,\Omega_{\mathrm{Y}}\mathbf{Y} + \mathbf{N}_{\mathrm{Y}}(\mathbf{Y},\mathbf{Z}) = 0$$
$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} + i\,\Omega_{\mathrm{Z}}\mathbf{Z} + \mathbf{N}_{\mathrm{Z}}(\mathbf{Y},\mathbf{Z}) = 0$$

The slow equations are formed by setting the tendencies of the fast components to zero (Daley, 1980, Lynch, 1989):

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} + i\,\Omega_{\mathrm{Y}}\mathbf{Y} + \mathbf{N}_{\mathrm{Y}}(\mathbf{Y},\mathbf{Z}) = 0$$
$$i\,\Omega_{\mathrm{Z}}\mathbf{Z} + \mathbf{N}_{\mathrm{Z}}(\mathbf{Y},\mathbf{Z}) = 0$$

We take the transform of the general equation

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + i\,\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0$$

L is a linear operator. N is a nonlinear vector function.

- Transform analytically
- Diagonalize the system
- Invert analytically using $\mathfrak{L}^*$

$$\mathbf{Y}^{n+1} = \mathbf{Y}^n \exp(-2i\Omega_{\mathrm{Y}}\Delta t) - (i\Omega_{\mathrm{Y}})^{-1}\mathbf{N}_{\mathrm{Y}}^n[1 - \exp(-2i\Omega_{\mathrm{Y}}\Delta t)]$$
$$\mathbf{Z}^{n+1} = -(i\Omega_{\mathrm{Z}})^{-1}\mathbf{N}_{\mathrm{Z}}^n$$

So, $\mathbf{Y}^{n+1}$ is the analytical solution at time $(n+1)\Delta t$ for $\mathbf{N}_{\mathrm{Y}}$ constant, and Z satisfies a balance equation.

These equations correspond essentially to Daley’s (1980) Scheme B.

There is a close relationship between the Laplace transform scheme and Daley’s filtered scheme.

The slow components in Daley’s Scheme B are calculated by a leapfrog method.

For the LT scheme, they are analytical solutions (for constant N_Y).

A Reasonable Question

If we simply return to the time domain, why bother with the Laplace Transform at all?

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Because it provides guidance and insight!

By analogy, consider a time filter:

$$y_n = \sum_{h=-N}^{+N} a_h \, x_{n-h}$$

This is defined completely in the time domain;

But it is greatly illuminated by considering the response in the frequency domain.

LT scheme in the STSWM Eulerian model

The transformed spectral equations are:

$$\widehat{\eta_\ell^m} = \frac{1}{s}\{\eta_\ell^m\}^{n-1} + \frac{1}{s^2}\{\mathcal{N}_\ell^m\}^n$$

$$\widehat{\delta_\ell^m} = d\left(\mathcal{R} + \frac{1}{s}\frac{\ell(\ell+1)}{a^2}\mathcal{Q}\right)$$

$$\widehat{\Phi_\ell^m} = d\left(\mathcal{Q} - \frac{1}{s}\bar{\Phi}^*\mathcal{R}\right)$$

where the poles of d, $\mathcal{Q}$ and $\mathcal{R}$ are known.
For example, $d = s^2/(s^2+\omega_\ell^2)$.

By inspection, we can apply the analytical operator $\mathfrak{L}^*$ to obtain the solution.

The technicalities for the Lagrangian model are non-trivial

We must move back and forth between physical space and Hough space

In principle, it is straightforward;
In practice, it is intricate.

Details are in a paper in preparation.

Outline

Basic Theory

Numerical Inversion

Ordinary Differential Equations

Application to Numerical Weather Prediction

Kelvin Waves & Phase Errors

Lagrangian Formulation

Orographic Resonance

Analytical Inversion

Results

Results with Eulerian Model

In all Eulerian simulations:

- $\Delta t = 600$ s
- Spectral resolution T119
- No explicit diffusion included
- Normalised ℓ_∞ error measure
- Ref: Semi-implicit T213 with $\Delta t = 90$ s

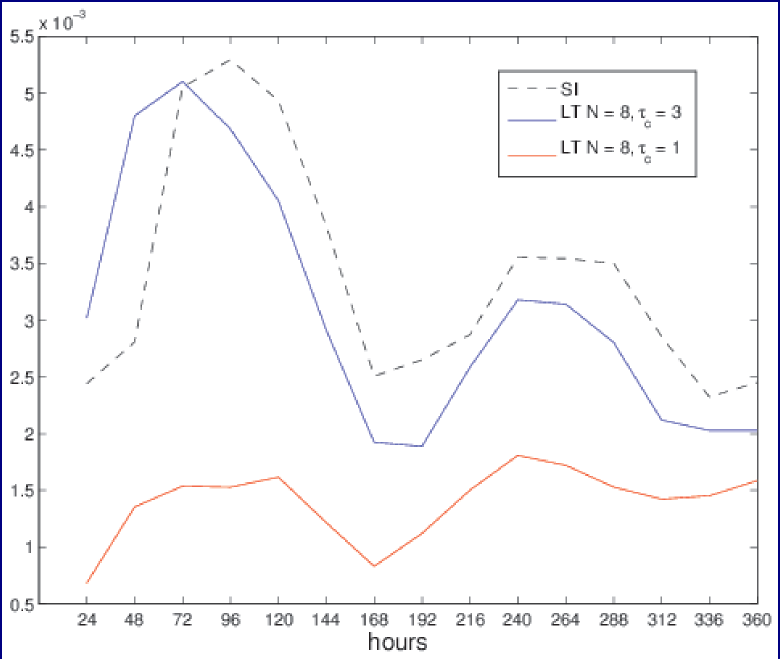
Eulerian Model: Mountain flow (Case 5)

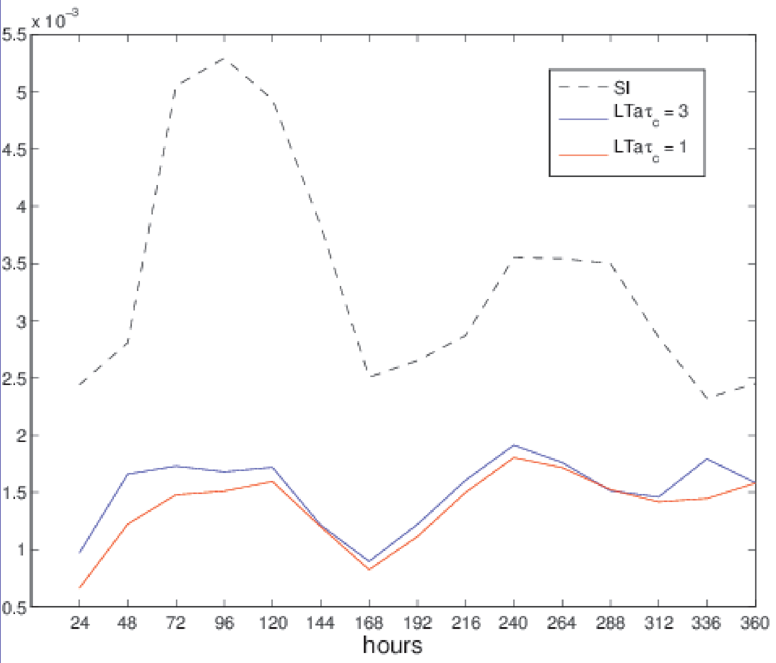
First plot:

- Reference SI scheme
- Numerical LT with N = 8, cutoff period 3 hours
- Numerical LT with N = 8, cutoff period 1 hour.

Second plot:

- Reference SI scheme
- Analytical LT with cutoff period 3 hours
- Analytical LT with cutoff period 1 hour.


x 10^-3
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
hours
SI
LT N = 8, τ_c = 3
LT N = 8, τ_c = 1



x 10^-3
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
24 48 72 96 120 144 168 192 216 240 264 288 312 336 360
hours
SI
LTaτc = 3
LTaτc = 1

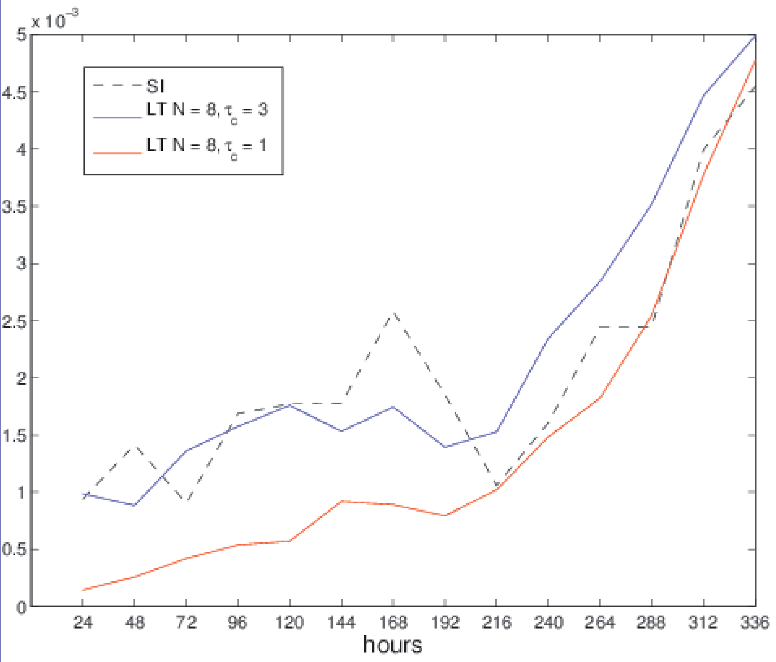
Eulerian Model: Rossby-Haurwitz (Case 6)

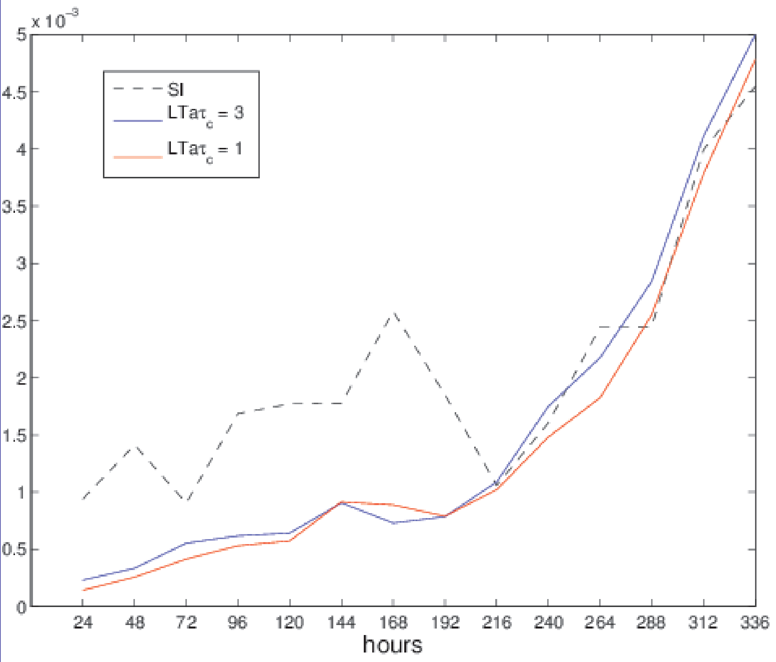
First plot:

- Reference SI scheme
- Numerical LT with N = 8, cutoff period 3 hours
- Numerical LT with N = 8, cutoff period 1 hour.

Second plot:

- Reference SI scheme
- Analytical LT with cutoff period 3 hours
- Analytical LT with cutoff period 1 hour.


x 10^-3
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
SI
LT N = 8, τ_c = 3
LT N = 8, τ_c = 1
24 48 72 96 120 144 168 192 216 240 264 288 312 336
hours



x 10^-3
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
SI
LTaτ_c = 3
LTaτ_c = 1
24
48
72
96
120
144
168
192
216
240
264
288
312
336
hours

Remarks

The truncation to N=8 has a big effect, particularly in Case 5. The choice of cutoff period is important.

Clearly there are motions of frequency between one and three hours that are being damaged and damped

In the analytic case, this isn't an issue:
the two integrations match closely.

Results with Lagrangian Model

In all Lagrangian simulations:

- Cutoff period of 1 hour
- Spectral resolution T119
- Normalised ℓ_∞ error measure
- SETTLS treatment of the rhs nonlinear terms
- Back trajectories: McGregor scheme (MWR 1993)
- Reference: Semi-implicit T213 with $\Delta t = 90\,\mathrm{s}$

[Also run with trajectory scheme of GEM model].

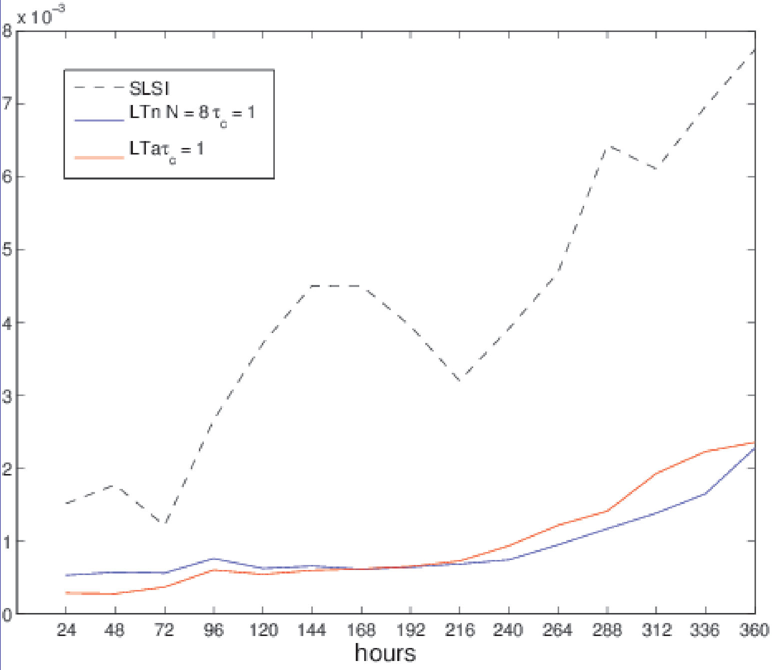
Lagrange Model: Mountain flow (Case 5)

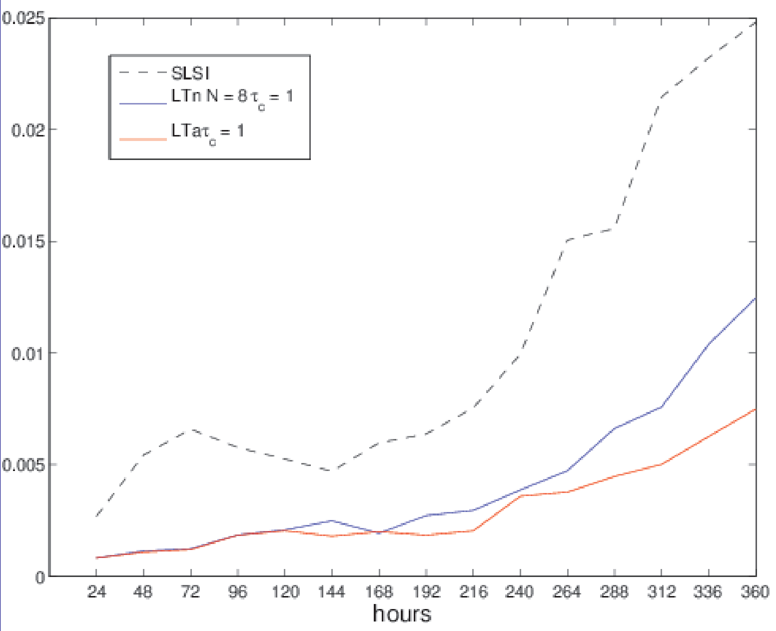
Both plots:

- Numerical LT with N = 8
- Analytical LT
- Cutoff period: 1 hour in both cases
- Reference: Semi-implicit scheme.

Time steps

- First plot: $\Delta t = 600\,\mathrm{s}$
- Second plot: $\Delta t = 1800\,\mathrm{s}$


x 10^-3
8
7
6
5
4
3
2
1
0
SLSI
LTn N = 8 τ_c = 1
LTaτ_c = 1
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
hours


SLSI
LTn N = 8 τ_c = 1
LTa τ_c = 1
0.025
0.02
0.015
0.01
0.005
0
24
48
72
96
120
144
168
192
216
240
264
288
312
336
360
hours

Remarks

In the Lagrangian case, there is more potential for error, with Hough mode transformations and numerical inversion of matrices.

The Rossby wave case looks terrible!

We make approximations of the form

$$\widehat{f\,\zeta} = f\,\widehat{\zeta}\,, \qquad \widehat{\beta\,u} = \beta\,\widehat{u}$$

These may be insufficiently accurate.

We are investigating this issue.

Conclusion

Old Results

- LT scheme effectively filters HF waves
- LT scheme more accurate than SI scheme
- LT scheme has no orographic resonance.

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Old Results

- LT scheme effectively filters HF waves
- LT scheme more accurate than SI scheme
- LT scheme has no orographic resonance.

New Results

- Analytical LT more accurate than numerical
- Lagrangian scheme: more work needed
- Problems remain with Coriolis terms.

Thank you