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Energy Spectra from Entropy Principles

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Introduction

The energy distribution in turbulent systems varies widely.

A power-law dependence on wavenumber is common:

- **Burgers turbulence (1D): a K^{-2} spectrum**
- **Atmosphere, synoptic range (∼2D): a K^{-3} spectrum**
- **Fully developed 3D turbulence: $K^{-5/3}$ spectrum.**

Canonical Statistical Mechanics

The equilibrium statistical mechanics of classical systems is based on Liouville's Theorem.

This theorem continues to hold under spectral truncation.

As a result, the probability distribution function (PDF) of a constant of the motion, K, has the canonical form

$$Z(\beta)\exp(-\beta K)$$

The *partition function* $Z(\beta)$ normalizes the distribution.

The quantity $1/\beta$ plays a role analogous to the temperature in thermodynamic systems.

Non-equilibrium steady state

Typically, turbulent motions are far from equilibrium.

Turbulence is a dissipative, irreversible process.

It is often stated that equilibrium statistical mechanics is inapplicable to turbulence.

However, if forcing and dissipation are on average in balance, a non-equilibrium steady state may be reached.

We consider driven and damped motions in two dimensions, in which the mean forcing and damping are in balance.

Applications of the theory

Using the balance between forcing and damping as constraints, we derive a range of energy spectra for such non-equilibrium systems.

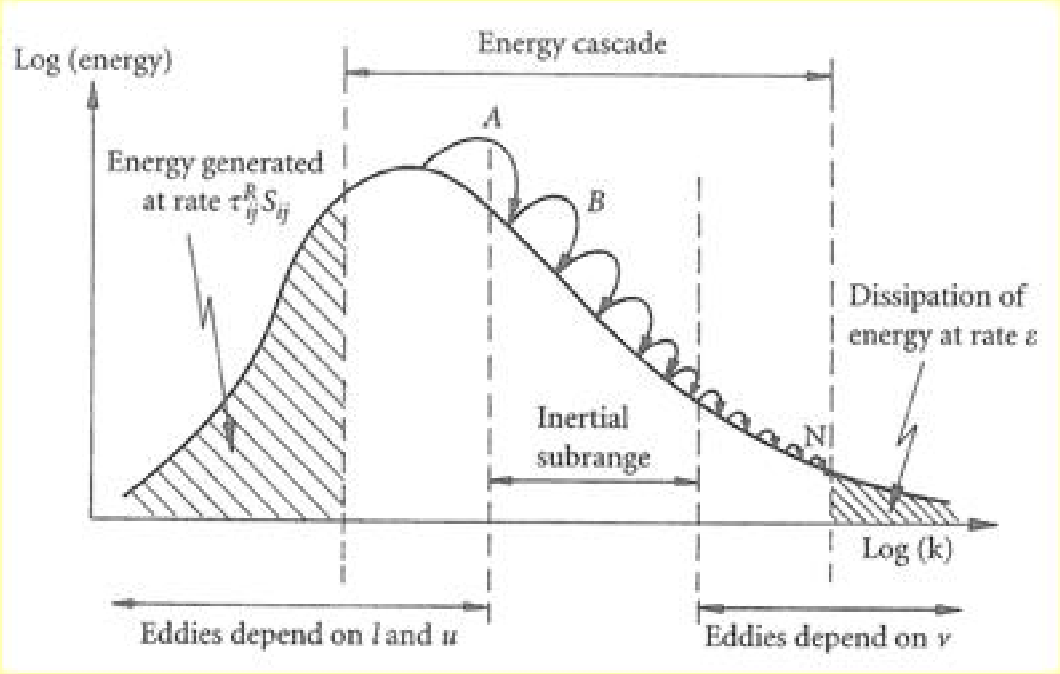
- **Burgers' Equation:** K^{-2}
- **2D turbulence on bi-periodic domain:** K^{-3}
- **Geostrophic turbulence (∼2D) on the sphere:** K^{-3}

We compare theoretical results with numerical integrations.

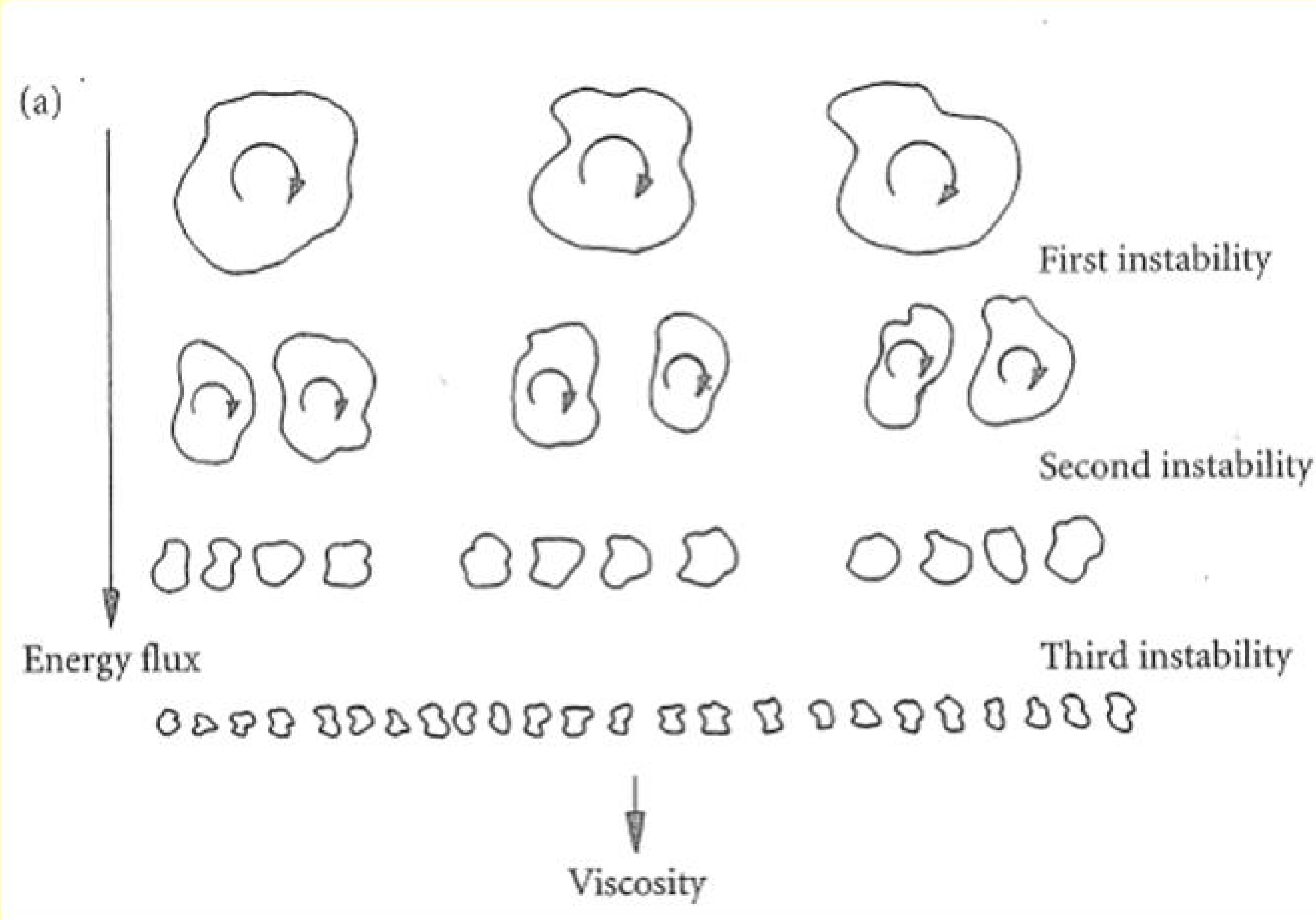
Good agreement is found.

Background

L F Richardson



"Big whirls have little whirls … "


(a)
First instability
Second instability
Energy flux
Third instability
Viscosity

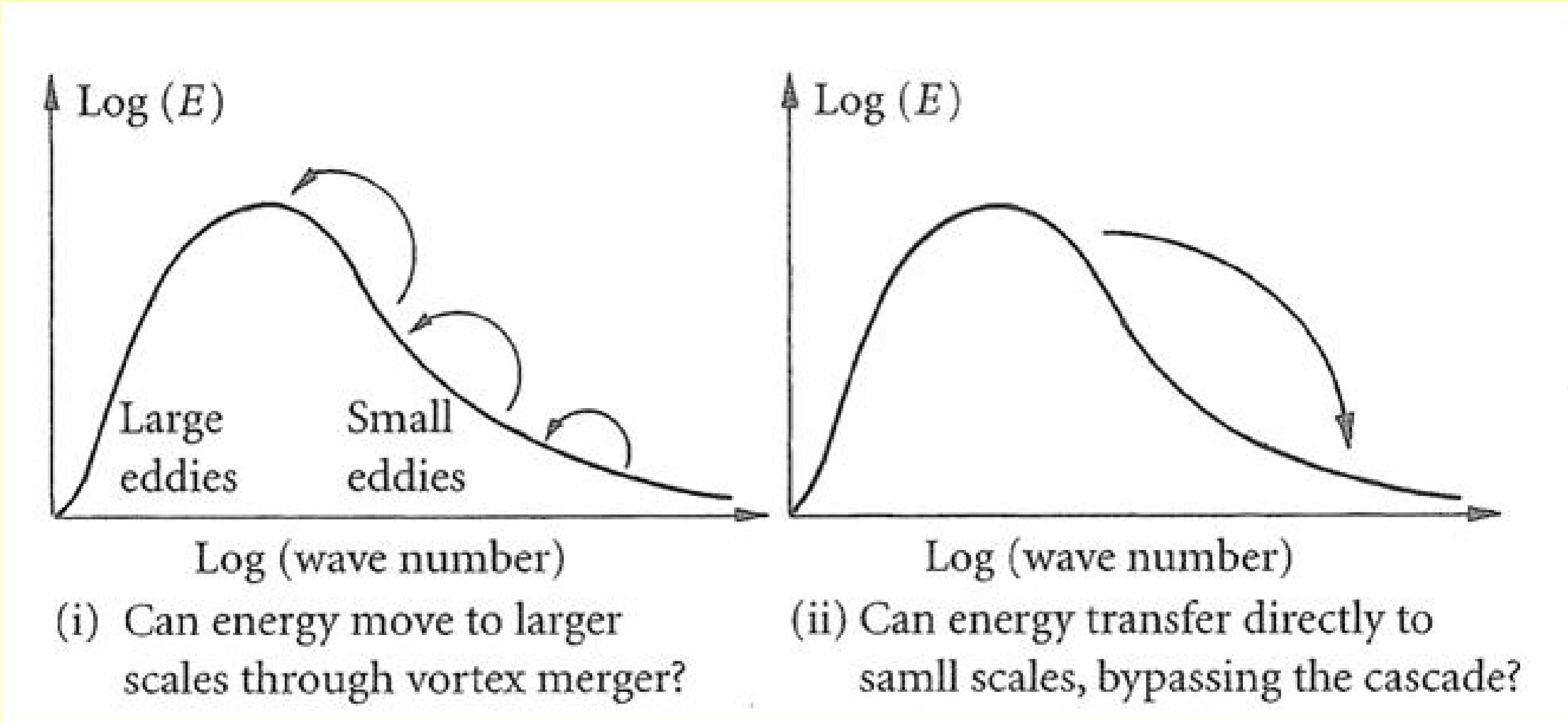


Figure from Davidson: *Turbulence*

The Entropy

The state of the system is

$$\mathbf{a} = (a_1, a_2, \ldots, a_N) \, .$$

The distribution function is

$$W = W(\mathbf{a})$$

We wish to find W.

We define the entropy:

$$S = -\int W \log W \, d\mathbf{a}$$

We seek the W that maximizes S subject to constraints.

Lagrange Multipliers

Since W is a PDF we have

$$\int W(\mathbf{a})\,d\mathbf{a} = 1\,.$$

Consider a constraint on the expected value of $\mathcal{K}(\mathbf{a})$:

$$\langle\mathcal{K}\rangle = \int \mathcal{K}(\mathbf{a})W(\mathbf{a})\,d\mathbf{a} = \mathcal{K}_0\,.$$

We use Lagrange multipliers:

$$S_{\mathrm{constrained}} = S + \rho\left(\int W\,d\mathbf{a} - 1\right) + \lambda\Big(\mathcal{K}_0 - \langle\mathcal{K}\rangle\Big)\,.$$

The Canonical Distribution

The variational derivative, varying W, gives

$$- \log W - 1 + \rho - \lambda \mathcal{K} = 0$$

The canonical distribution function is

$$W = e^{\rho - 1} \exp(-\lambda \mathcal{K}) \,.$$

We will apply this to several specific cases.

Forced Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = F(x) + \nu\frac{\partial^2 u}{\partial x^2}$$

Periodicity on interval $x \in [0, 2\pi]$.

The energy principle is

$$\frac{dE}{dt} = \mathcal{F} - \mathcal{D}$$

where

$$E = \int \tfrac{1}{2}u^2\,dx \quad \mathcal{F} = \int Fu\,dx \quad \mathcal{D} = \nu\int u_x^2\,dx$$

Spectral Expansion

We expand $u(x,t)$ in spectral components (assuming u even).

$$u(x,t) = \sum_{k=1}^{N} a_k(t) \cos kx$$

The energy equation is, as above,

$$\frac{dE}{dt} = \mathcal{F} - \mathcal{D}$$

where

$$\mathcal{F} = \tfrac{1}{2} \textstyle\sum_{k=1}^{N} f_k a_k \qquad \mathcal{D} = \tfrac{1}{2}\nu \textstyle\sum_{k=1}^{N} k^2 a_k^2 \,.$$

No Forcing or Dissipation

We consider the case without forcing or dissipation.

Assume the expected value of the total energy is constant.

$$\langle E \rangle = \tfrac{1}{2} \textstyle\sum_{k=1}^{N} \langle a_k^2 \rangle = E_0 \,,$$

The distribution function is

$$W = e^{\rho - 1} \exp(-\lambda E) \,.$$

We make this more explicit:

$$W = e^{\rho - 1} \exp\left(-\lambda \pi \sum_{k=1}^{N} a_k^2 \right) \,.$$

Repeating:

$$W = e^{\rho-1} \exp\left(-\lambda\pi \sum_{k=1}^{N} a_k^2\right) .$$

Defining $\sigma^2 = 1/(2\lambda\pi)$, we have

$$W = \prod_{k=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{a_k^2}{2\sigma^2}\right) = \prod_{k=1}^{N} \mathcal{N}(0, \sigma, a_k) .$$

Each component a_k, has zero mean.

Each has the same standard deviation σ.

There is equi-partitioning of energy.

Balance: ⟨Forcing⟩ = ⟨Dissipation⟩

We now assume that

$$\left\langle \frac{dE}{dt} \right\rangle = \left\langle \tfrac{1}{2} \textstyle\sum_{k=1}^{N} \left(f_k a_k - \nu k^2 a_k^2 \right) \right\rangle = 0 \, .$$

Define the mean and variance

$$\mu_k = f_k/(2\nu k^2) \quad \sigma_k^2 = 1/(2\lambda\nu k^2)$$

Then

$$W = \prod_{k=1}^{N} \mathcal{N}(\mu_k, \sigma_k, a_k) \, .$$

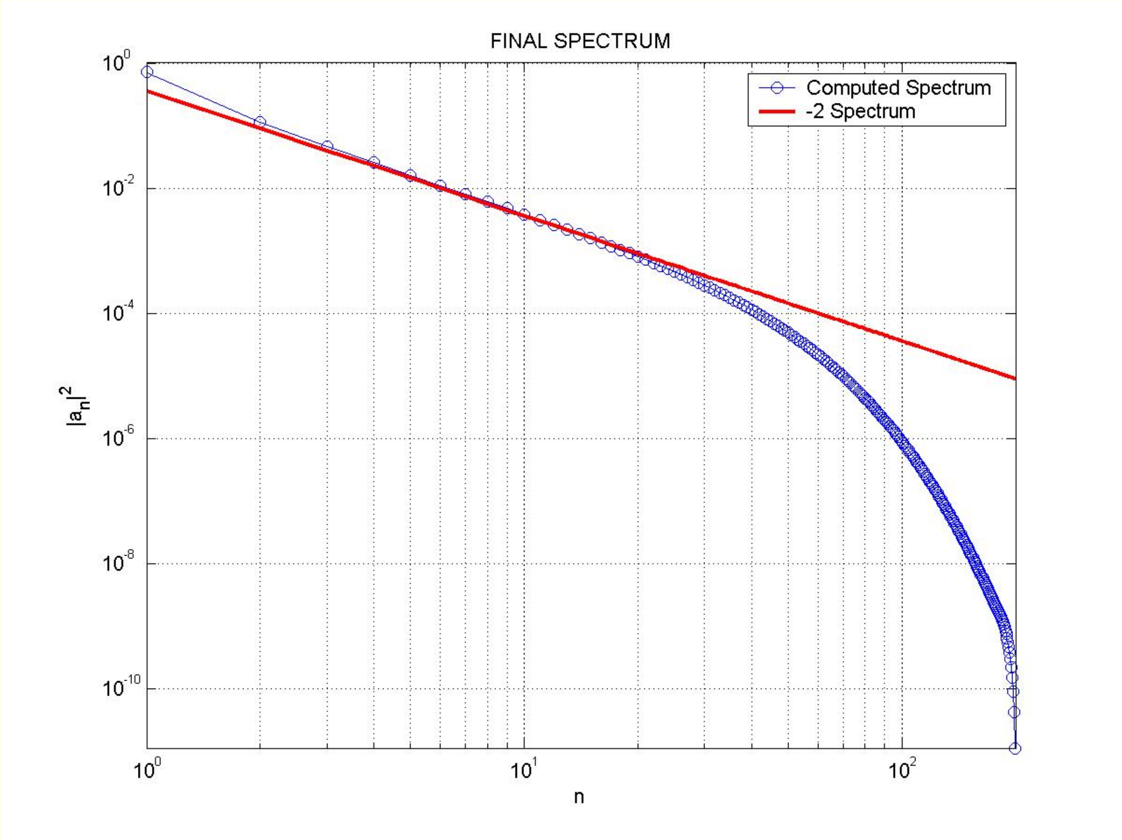
The mean and variance are different for each component.

$$\mu_k = f_k/(2\nu k^2) \quad \sigma_k^2 = 1/(2\lambda\nu k^2)$$

Repeat: the mean and variance are

$$\mu_k = f_k/(2\nu k^2) \quad \sigma_k^2 = 1/(2\lambda\nu k^2)$$

This implies a k^{-2} spectrum . . .



Energy spectrum for Burgers' equation

Two-dimensional Turbulence

We now consider turbulent motion on a two-dimensional bi-periodic domain $[0, 2\pi] \times [0, 2\pi]$.

The vorticity equation is

$$\frac{\partial \omega}{\partial t} + J(\psi, \omega) = F + \nu \nabla^2 \omega \, .$$

Both total energy and total enstrophy are conserved for unforced and undamped flow.

$$\frac{dE}{dt} = \frac{d}{dt}\iint \tfrac{1}{2}\nabla\psi\cdot\nabla\psi\,dxdy = \iint \left[F\psi - \nu\omega^2\right]dxdy$$

$$\frac{dS}{dt} = \frac{d}{dt}\iint \tfrac{1}{2}\omega^2\,dxdy = \iint \left[F\omega - \nu\nabla\omega\cdot\nabla\omega\right]dxdy\,.$$

The spectral enstrophy equation is

$$\frac{dS}{dt} = \frac{d}{dt}\sum_{k\ell} \tfrac{1}{2}\omega_{k\ell}^2 = \sum_{k\ell}\left[f_{k\ell}\omega_{k\ell} - \nu(k^2+\ell^2)\omega_{k\ell}^2\right]$$

We now assume

$$\left\langle \frac{dS}{dt} \right\rangle = \left\langle \sum_{k\ell} \left(f_{k\ell}\omega_{k\ell} - \nu K^2 \omega_{k\ell}^2 \right) \right\rangle = 0\,,$$

where $K^2 = k^2 + \ell^2$**.**

We define mean and variance

$$\mu_{k\ell} = f_{k\ell}/(2\nu K^2) \quad \sigma_{k\ell}^2 = 1/(2\lambda\nu K^2)\,.$$

The distribution function becomes

$$W = \prod_{k\ell} \mathcal{N}(\mu_{k\ell}, \sigma_{k\ell}, \omega_{k\ell})\,.$$

The variance $\sigma_{k\ell}^2$ has a K^{-2} dependence on wavenumber.

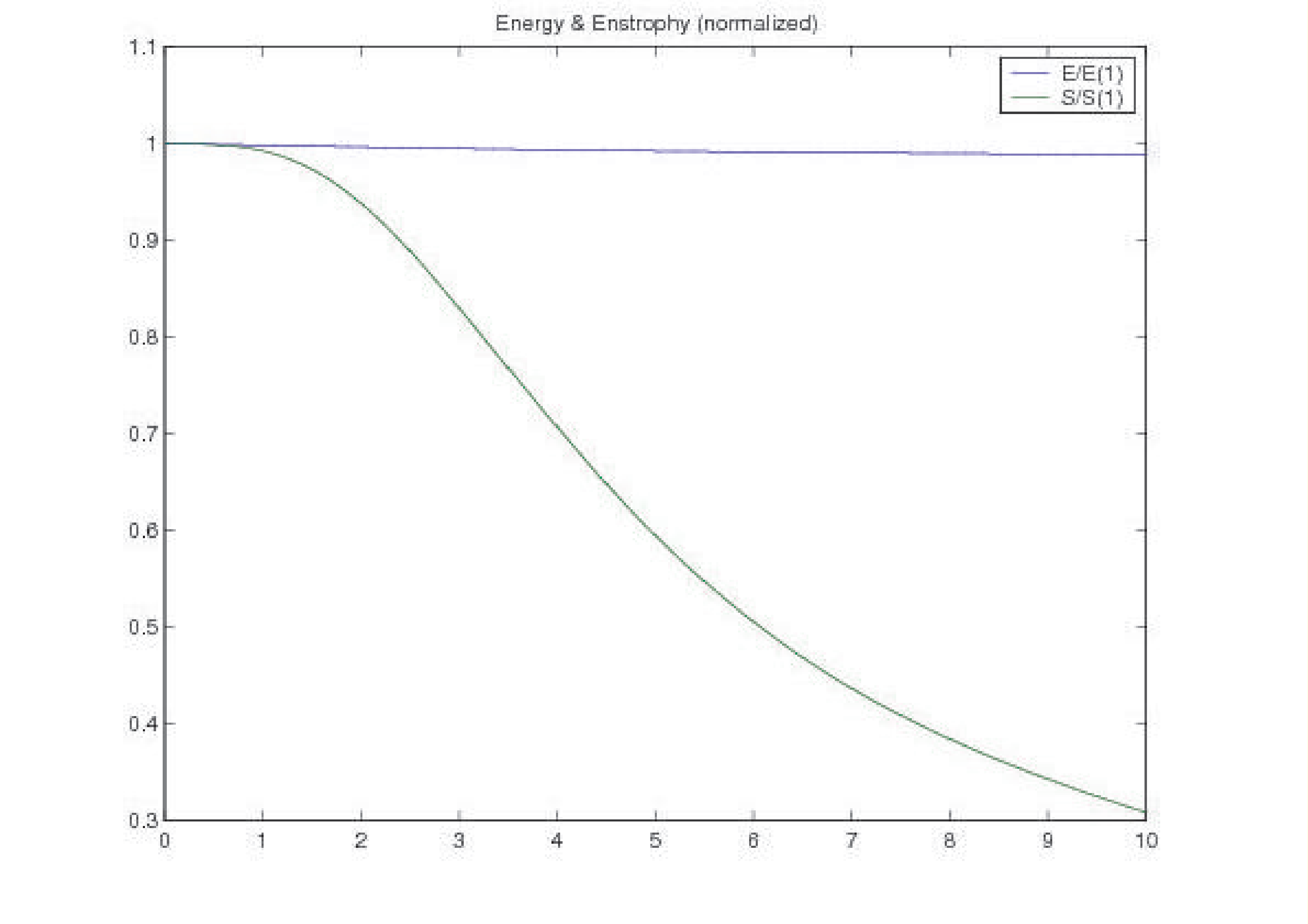
Thus, in the inertial range, we have

$$\langle\omega_{k\ell}^2\rangle \propto K^{-2}\,.$$

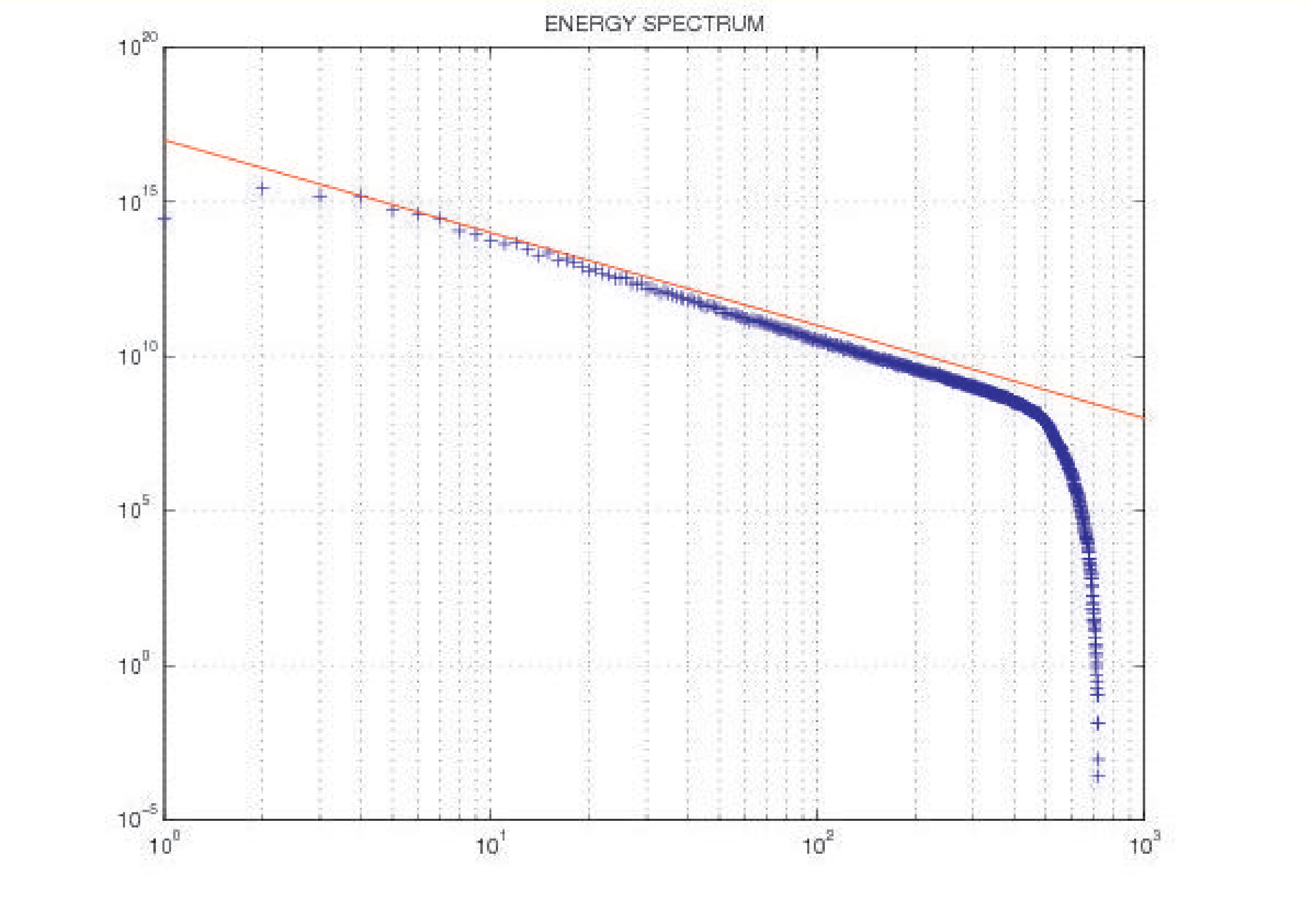
The energy spectrum (integrating over the angle) is

$$E_K \propto K^{-3}$$

This is precisely what we expect for the enstrophy cascade in two-dimensional turbulent flow.



Energy and Enstrophy development during decaying 2D quasi-geostrophic turbulence.



Energy spectrum after 10 days (solid line is K^{-3} spectrum).

The Barotropic PV Equation

We now introduce rotation and of vortex stretching.

With the usual quasi-geostrophic approximations, we have

$$\frac{\partial}{\partial t}(\omega - \Lambda\psi) + J(\psi, \omega) + \beta\frac{\partial \psi}{\partial x} = \mathcal{F} - \mathcal{D}\,,$$

where $\beta = df/dy$ **and** $\Lambda = f_0^2/gH$ **(Pedlosky, 1987).**

The energy equation is:

$$\frac{d}{dt}\iint \tfrac{1}{2}\left[(\nabla\psi)^2 + \Lambda\psi^2\right]\,dxdy = \iint \left[F\psi - \nu\omega^2\right]dxdy\,.$$

The enstrophy equation is

$$\frac{d}{dt}\iint \tfrac{1}{2}\,(\omega - \Lambda\psi)^2\,dxdy = \iint \left[F(\omega-\Lambda\psi) - \nu\nabla(\omega-\Lambda\psi)\cdot\nabla\omega\right]dxdy\,.$$

The enstrophy equation in spectral form:

$$\frac{dS}{dt} = \frac{d}{dt}\sum_{k\ell}\tfrac{1}{2}(\omega_{k\ell} - \Lambda\psi_{k\ell})^2 = \sum_{k\ell}\left(1 + \frac{\Lambda}{K^2}\right)\left[f_{k\ell}\omega_{k\ell} - \nu K^2\omega_{k\ell}\right]\,.$$

We assume that the forcing and damping balance:

$$\left\langle -\nu K^2 \left(1+\frac{\Lambda}{k^2}\right)\left[\omega_{k\ell}^2 - \frac{f_k}{\nu K^2}\omega_{k\ell}\right]\right\rangle = 0\,.$$

Defining $\mu_{k\ell} = f_{k\ell}/(2\nu K^2)$**, this constraint is** $\langle \mathcal{K} \rangle = 0$**, where**

$$\mathcal{K} = -\sum_{k\ell} \nu K^2 \left(1+\frac{\Lambda}{K^2}\right)\left[(\omega_{k\ell}-\mu_{k\ell})^2 - \mu_{k\ell}^2\right]$$

The probability density function may once again be written in normal form, with variance

$$\sigma_{k\ell}^2 = \frac{1}{2\lambda\nu(K^2+\Lambda)}\,.$$

Outside the forcing region, where $\mu_{k\ell} = 0$**, we have**

$$\langle \omega_{k\ell}^2 \rangle \propto \frac{1}{K^2+\Lambda}\,.$$

The spectrum of the vorticity components is

$$\omega_{k\ell} \propto \frac{1}{\sqrt{K^2 + \Lambda}}$$

Thus, the streamfunction goes as

$$\psi_{k\ell} \propto \frac{1}{K^2\sqrt{K^2 + \Lambda}}$$

and the components of energy vary as

$$E_{k\ell} \propto \frac{1}{K^2(K^2 + \Lambda)}$$

Summing over all components having total wavenumber K,

$$E_K \propto \frac{1}{K(K^2 + \Lambda)} .$$

For $K \ll \Lambda$ we get a K^{-1} spectrum.
For $K \gg \Lambda$ we get a K^{-3} spectrum.

Geostrophic Turbulence on Sphere

The barotropic vorticity equation is

$$\frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = F + \sum_p (-1)^{p+1} \nu_p \nabla^{2p+2} \psi,$$

where q is the quasi-geostrophic potential vorticity

$$q = f + \omega - \Lambda\psi + f\frac{h}{H}.$$

(h is the orography).

The system is forced by F and damped by (hyper-)viscosity.

The system has energy and potential enstrophy principles:

$$\frac{dE}{dt} = \mathcal{F} - \mathcal{D},$$
$$\frac{dZ}{dt} = \mathcal{G} - \mathcal{H},$$

where

$$E = \frac{1}{4\pi}\iint \tfrac{1}{2}[\mathbf{v}^2 + \Lambda\psi^2]\,dS, \qquad Z = \frac{1}{4\pi}\iint \tfrac{1}{2}q^2\,dS,$$

$$\mathcal{F} = -\frac{1}{4\pi}\iint \psi F\,dS, \qquad \mathcal{D} = \frac{1}{4\pi}\iint \psi\left[\sum_p (-1)^{p+1}\nu_p \nabla^{2p+2}\psi\right]\,dS,$$

$$\mathcal{G} = \frac{1}{4\pi}\iint qF\,dS, \qquad \mathcal{H} = -\frac{1}{4\pi}\iint q\left[\sum_p (-1)^{p+1}\nu_p \nabla^{2p+2}\psi\right]\,dS.$$

The vorticity ω is expanded in spherical harmonics

$$\omega(\lambda, \phi, t) = \sum_{mn} \omega_{mn}(t) Y_{mn}(\lambda, \phi),$$

The spectral coefficients are given by

$$\omega_{mn}(t) = \frac{1}{4\pi} \int Y_{mn}(\lambda, \phi) \omega(\lambda, \phi, t)\, dS.$$

They are related to the coefficients of ψ by

$$\omega_{mn} = -e_n \psi_{mn} \qquad \textbf{where} \qquad e_n = n(n+1)\,.$$

The field $f + f(h/H)$ is expanded as

$$f + f\frac{h}{H} = \sum_{mn} f_{mn} Y_{mn},$$

Then

$$q_{mn} = f_{mn} - \varepsilon_n \psi_{mn}\,, \qquad \textbf{where} \qquad \varepsilon_n = \Lambda + e_n\,.$$

The damping term may be written as

$$\sum_p (-1)^{p+1} \nu_p \nabla^{2p+2} \psi = \sum_{mn} \left(\sum_p \nu_p e_n^{p+1} \right) \psi_{mn} Y_{mn}$$

With these definitions we may write, in terms of ψ_{mn}:

$$E = \sum_{mn} \tfrac{1}{2}\varepsilon_n \psi_{mn}^2, \qquad Z = \textstyle\sum_{mn} \left[\tfrac{1}{2}\varepsilon_n^2 \psi_{mn}^2 - \varepsilon_n f_{mn} \psi_{mn} + \tfrac{1}{2} f_{mn}^2\right],$$

$$\mathcal{F} = \sum_{mn} -F_{mn}\psi_{mn}, \qquad \mathcal{D} = \sum_{mn} d_n \psi_{mn}^2,$$

$$\mathcal{G} = \sum_{mn} \left[-\varepsilon_n F_{mn} \psi_{mn} + f_{mn} F_{mn}\right], \quad \mathcal{H} = \sum_{mn} \left[d_n \varepsilon_n \psi_{mn}^2 - d_n f_{mn} \psi_{mn}\right].$$

These quantities are formally similar to previous cases.

The Unforced-Undamped Case

The constraints to be used when there is neither forcing nor damping are constancy of energy E and pot. enstrophy Z:

$$\langle \mathcal{K}_1 \rangle = \langle E \rangle = E^0, \;\; \langle \mathcal{K}_2 \rangle = \langle Z \rangle = Z^0,$$

For the multivariate normal distribution function we get:

$$W(\psi_{-N,-N}, ..., \psi_{N,N}) = \prod_{mn} \mathcal{N}(\mu_{mn}, \sigma_{mn}, \psi_{mn}),$$

where the variance and mean are

$$\sigma_{mn}^2 = \frac{1}{\varepsilon_n(\beta + \gamma\varepsilon_n)} \qquad \textbf{and} \qquad \mu_{mn} = \frac{\gamma f_{mn}}{\beta + \gamma\varepsilon_n}\,.$$

The mean values are determined by the Coriolis parameter and the orography and the variance depends on ε_n.

We eventually get the energy and esntrophy spectra

$$E_n = \frac{2n+1}{2(\beta+\gamma\varepsilon_n)} + \frac{\gamma^2\varepsilon_n}{2(\beta+\gamma\varepsilon_n)^2}\sum_{m=-n}^{n} f_{mn}^2 \,,$$

$$Z_n = \frac{(2n+1)\varepsilon_n}{2(\beta+\gamma\varepsilon_n)} + \frac{\beta^2}{2(\beta+\gamma\varepsilon_n)^2}\sum_{m=-n}^{n} f_{mn}^2 \,.$$

These expressions are very similar to those for two-dimensional turbulence.

When the Coriolis parameter and the orography are zero E_n and Z_n behave as n^{-1} and n.

The Forced-Damped Case

We assume the system is in a statistically stationarity state. This means that we will use as constraints:

$$\langle \mathcal{K}_1 \rangle = \langle E \rangle = E_0, \;\; \langle \mathcal{K}_2 \rangle = \langle \mathcal{D} - \mathcal{F} \rangle = 0, \;\; \langle \mathcal{K}_3 \rangle = \langle \mathcal{H} - \mathcal{G} \rangle = 0.$$

This eventually gives the distribution function

$$W(\psi_{-N,-N}, ..., \psi_{N,N}) = \prod_{mn} \mathcal{N}(\mu_{mn}, \sigma_{mn}, \psi_{mn}),$$

with variance and mean

$$\sigma_{mn}^2 = \frac{1}{2d_n(\beta + \gamma\varepsilon_n)} \qquad \textbf{and} \qquad \mu_{mn} = \frac{\gamma f_{mn}}{2(\beta + \gamma\varepsilon_n)} - \frac{F_{mn}}{2d_n}.$$

For the energy and potential enstrophy we now have:

$$\langle E \rangle = \sum_{mn} \left[\frac{\varepsilon_n}{4d_n(\beta + \gamma\varepsilon_n)} + \frac{\gamma^2 \varepsilon_n f_{mn}^2}{8(\beta + \gamma\varepsilon_n)^2} + \frac{\varepsilon_n F_{mn}^2}{8d_n^2} - \frac{\gamma\varepsilon_n f_{mn} F_{mn}}{4d_n(\beta + \gamma\varepsilon_n)} \right],$$

$$\langle Z \rangle = \sum_{mn} \left[\frac{\varepsilon_n^2}{4d_n(\beta + \gamma\varepsilon_n)} + \frac{(\gamma\varepsilon_n + 2\beta)^2 f_{mn}^2}{8(\beta + \gamma\varepsilon_n)^2} + \frac{\varepsilon_n^2 F_{mn}^2}{8d_n^2} + \frac{\varepsilon_n(\gamma\varepsilon_n + 2\beta) f_{m}}{4d_n(\beta + \gamma\varepsilon_{}}\right.$$

We may write the result as

$$\langle E \rangle = \sum_{n=1}^{N} E_n, \quad \langle Z \rangle = \sum_{n=1}^{N} Z_n,$$

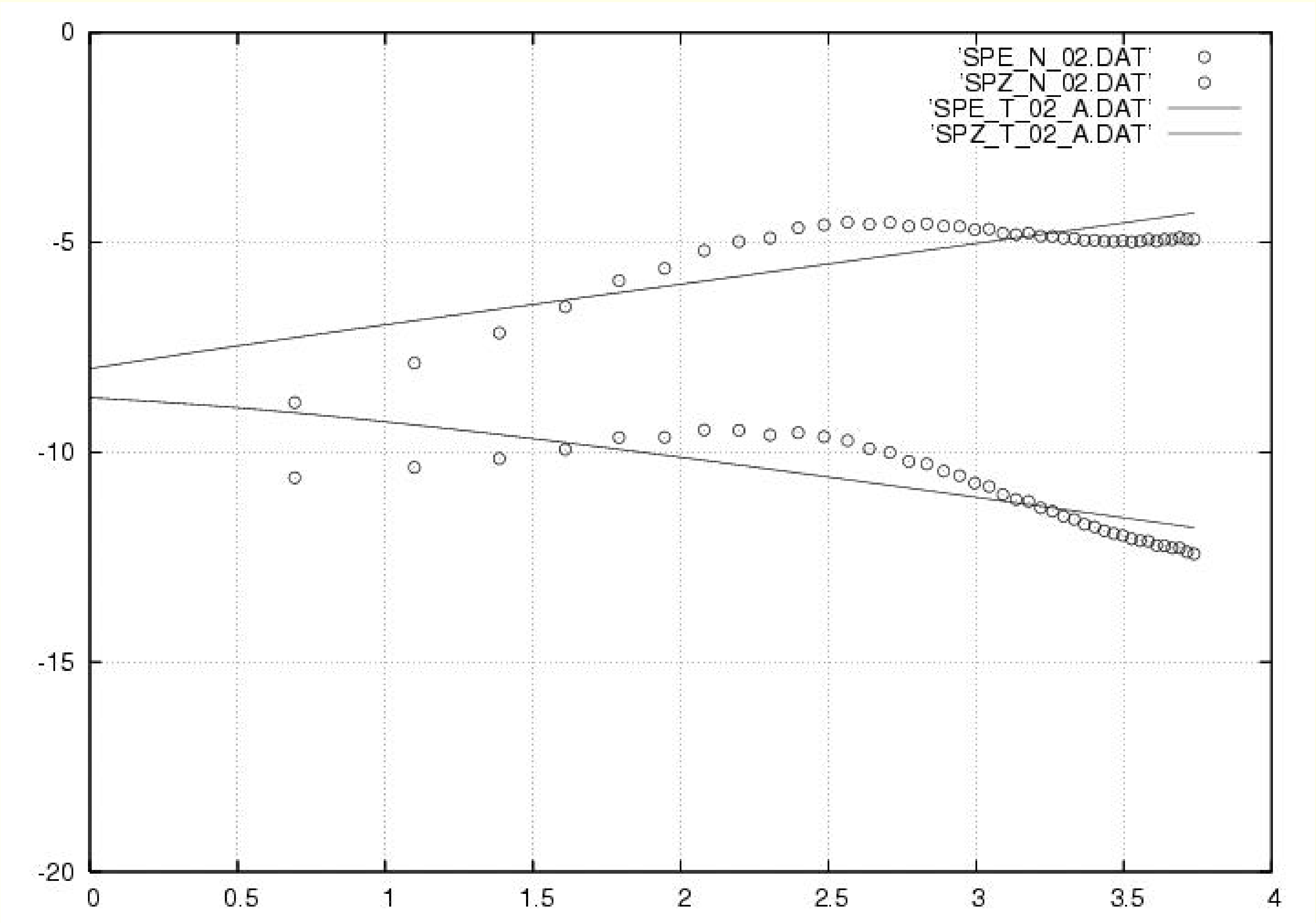
Since both ε_n and d_n are independent of m, we have

$$E_n = \frac{(2n+1)\varepsilon_n}{4d_n(\beta+\gamma\varepsilon_n)} + \sum_{m=-n}^{n} \varepsilon_n \frac{[\gamma f_{mn} - ((\beta+\gamma\varepsilon_n)F_{mn})/d_n]^2}{8(\beta+\gamma\varepsilon_n)^2},$$

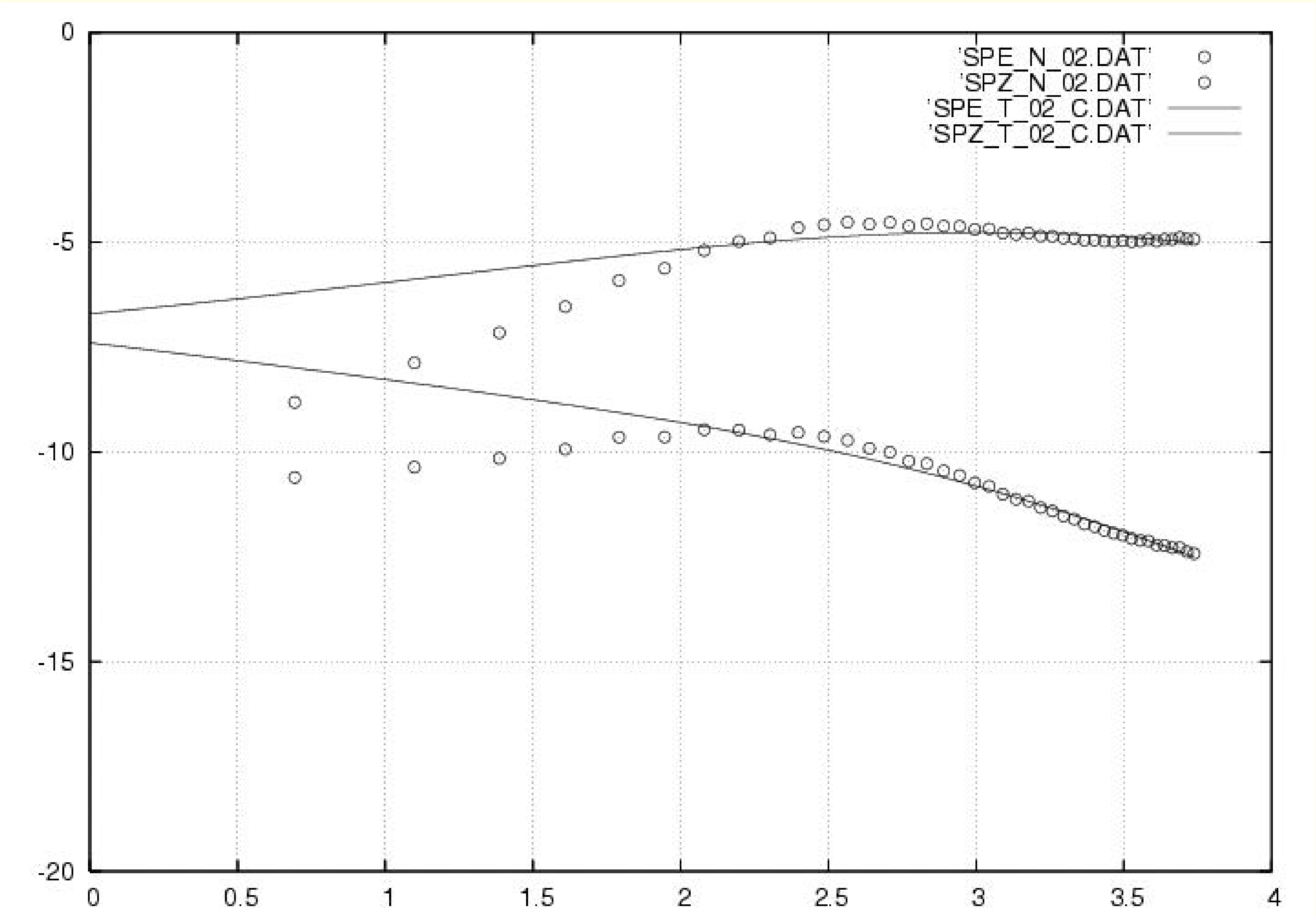
$$Z_n = \frac{(2n+1)\varepsilon_n^2}{4d_n(\beta+\gamma\varepsilon_n)} + \sum_{m=-n}^{n} \frac{[(\gamma\varepsilon_n+2\beta)f_{mn} + ((\beta+\gamma\varepsilon_n)\varepsilon_n F_{mn})/d_n]^2}{8(\beta+\gamma\varepsilon_n)^2}.$$

The extra terms arise because of the presence of the Earth's rotation, orography and forcing.

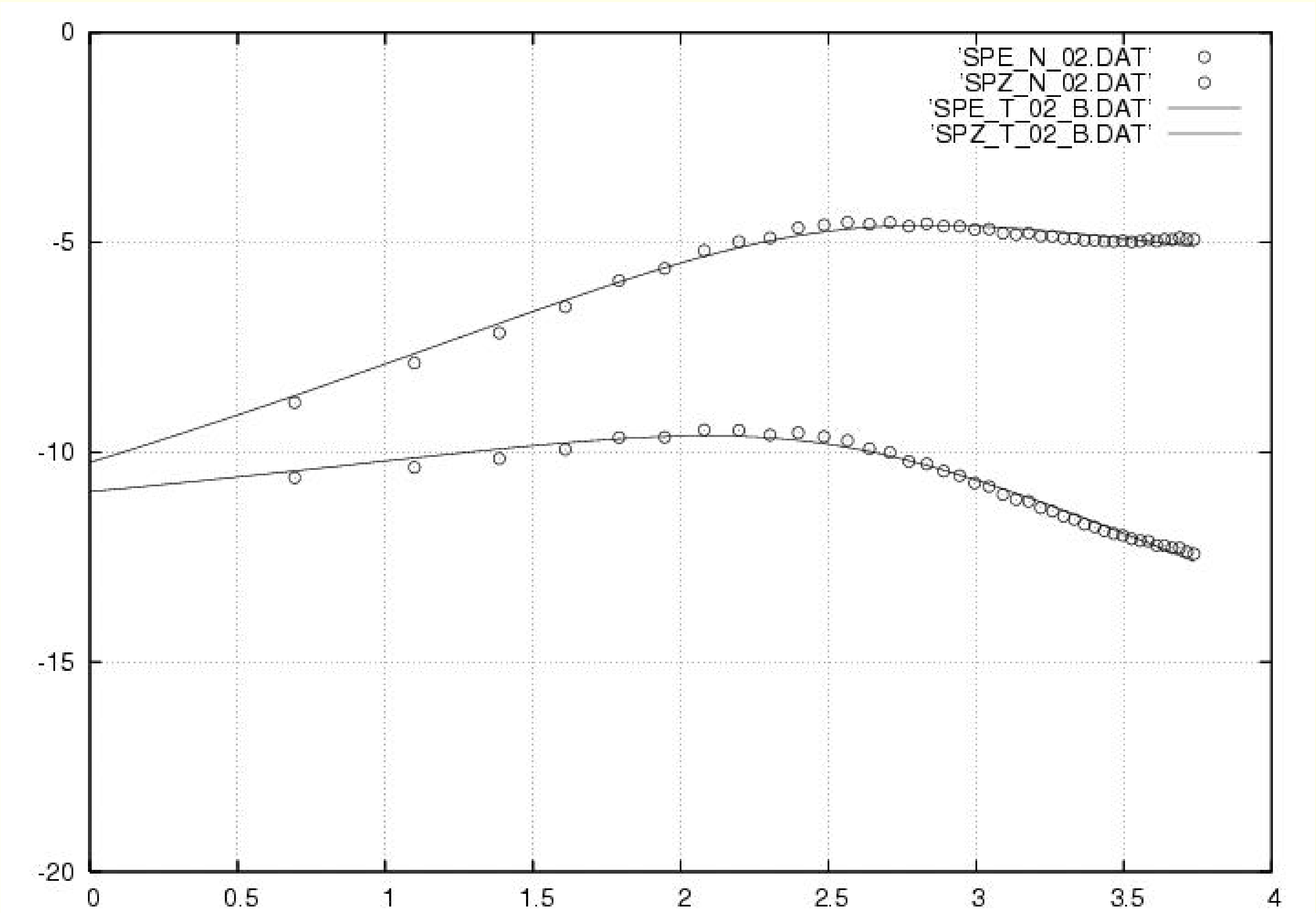
We will compare these spectra to numerical results.



Energy and Enstrophy constrained.



Energy and Enstrophy decay rates constrained.



Energy and Enstrophy and decay rates constrained.

The End

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