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Nonholonomic Noetherian Symmetries and Integrals of the Routh Sphere and the Chaplygin Ball

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Abstract—The dynamics of a spherical body with a non-uniform mass distribution rolling on a plane were discussed by Sergey Chaplygin, whose 150th birthday we celebrate this year. The Chaplygin top is a non-integrable system, with a colourful range of interesting motions. A special case of this system was studied by Edward Routh, who showed that it is integrable. The Routh sphere has a centre of mass offset from the geometric centre, but it has an axis of symmetry through both these points, and equal moments of inertia about all axes orthogonal to the symmetry axis. There are three constants of motion: the total energy and two quantities involving the angular momenta.

It is straightforward to demonstrate that these quantities, known as the Jellett and Routh constants, are integrals of the motion. However, their physical significance has not been fully understood. In this paper, we show how the integrals of the Routh sphere arise from Emmy Noether's invariance identity. We derive expressions for the infinitesimal symmetry transformations associated with these constants. We find the finite version of these symmetries and provide their geometrical interpretation.

As a further demonstration of the power and utility of this method, we find the Noetherian symmetries and corresponding integrals for a system introduced recently, the Chaplygin ball on a rotating turntable, confirming that the known integrals are directly obtained from Noether's theorem.

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INTRODUCTION

This paper is a contribution to the celebration of the 150th birthday of Sergey Alexeyevich Chaplygin (1869–1942), a renowned Russian physicist, mathematician, and mechanical engineer. Amongst many other topics, Chaplygin studied the dynamics of a sphere rolling on a plane. For this *Chaplygin top*, the mass distribution is eccentric, the three moments of inertia are distinct, and the geometric centre does not, in general, lie on any of the principal axes.

A special case of this system was studied by Edward Routh [17]. The Routh sphere is a spherical body with a non-uniform distribution of mass, free to roll without slipping on a plane surface. Its centre of mass is offset from the geometric centre, but it has an axis of symmetry through both these points, and equal moments of inertia about all axes orthogonal to the symmetry axis. This distinguishes it from the more general case studied by Chaplygin [6].

Routh [17] showed that the Routh sphere has two constants of motion in addition to the energy, and is an integrable system. The integrals or constants of motion, known as Jellett's constant and Routh's constant, have been treated in many studies. We mention, in particular, the important contributions [4, 7, 11, 12]. A simple proof that Jellett's and Routh's constants are integrals of the motion is given in Gray and Nickel [10]. However, as remarked by these authors, "The precise

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physical significance of the Routh constant remains elusive ... [and] it might be useful to try to find a direct connection between this constant of the motion and the underlying symmetries of the system" [10, p. 826]. This explicit connection is established in the present work.

Emmy Noether discovered a fundamental connection between symmetries or invariances of dynamical systems and conserved quantities or integrals of the motion. For a historical review, see [13]. In her seminal paper [16], Noether derived an identity valid whenever the action of the system has an invariance. In the case of extremal flow, in which the Euler-Lagrange or d'Alembert–Lagrange equations apply, this leads to a Noetherian conservation law. This is true both for systems with holonomic constraints and for systems with non-holonomic constraints that are linear in the velocities [1]. We will show in this paper how the integrals of the Routh sphere arise from Noether's invariance identity, and will derive expressions for the symmetry transformations associated with these constants.

As a further demonstration of the power and utility of Noether's theorem, we examine in § 5 the problem of the Chaplygin ball on a rotating turntable, recently studied in [2]. Using a systematic approach, we deduce the four known integrals and their associated symmetries directly from Noether's invariance identity.

1. THE INVARIANCE IDENTITY FOR NONHOLONOMIC SYSTEMS

The analysis below follows closely the seminal paper [1]. Associated with invariance of the action functional under transformations of the dependent and independent variables there is an identity, the *invariance identity*. We restrict ourselves, at the expense of generality but for simplicity of presentation, to the case when the transformation does not depend on the velocities. Then the invariance identity may be expanded in powers of the velocity variables $\dot{q}^\mu$, $\mu = 1, \ldots, N$, where N is the number of degrees of freedom, to yield a set of differential equations. If these can be solved, they provide the generators of a coordinate transformation that can be used to construct a constant of the motion.

For a dynamical system with a Lagrangian function, let us define the action functional

$$S = \int_{t_1}^{t_2} L\left(q(t), \frac{\mathrm{d}q(t)}{\mathrm{d}t}, t\right) \mathrm{d}t.$$

We consider a continuous transformation of the independent and dependent variables

$$t \to T(q, t; \alpha), \qquad q^\mu \to Q^\mu(q, t; \alpha),$$

where $\alpha \in \mathbb{R}$ is a free parameter. The case $\alpha = 0$ corresponds to the identity transformation, with $T(q, t; 0) = t$ and $Q^\mu(q, t; 0) = q^\mu$. We form the action S' using the new variables but the same functional form of the Lagrangian L:

$$S' = \int_{T_1}^{T_2} L\left(Q(T), \frac{\mathrm{d}Q(T)}{\mathrm{d}T}, T\right) \mathrm{d}T,$$

where $Q^\mu(T)$ (with slight abuse of notation) stands for the new dependent variable as a function of the new time. We consider the case where the action is invariant under the transformation: $S' = S$. For an infinitesimal perturbation, we write

$$\begin{aligned} q^\mu(t) \longrightarrow Q^\mu(T) &= q^\mu(t) + \epsilon\, \xi^\mu(q, t), \\ t \longrightarrow T &= t + \epsilon\, \tau(q, t). \end{aligned}$$

The coefficients of ϵ are called the generators of the transformation. They form the components of a vector field $(\xi^\mu(q, t), \tau(q, t))$, called an infinitesimal Noetherian symmetry. We expand the integrand of S' and express it as an integral with respect to t. Then the following invariance identity results:

$$\frac{\partial L}{\partial q^\mu} \xi^\mu + p_\mu \dot{\xi}^\mu + \frac{\partial L}{\partial t} \tau - H \dot{\tau} = 0, \tag{1.1}$$

where $p_\mu = \partial L/\partial\dot{q}^\mu$ is the conjugate momentum, the Hamiltonian is $H = p_\mu \dot{q}^\mu - L$, and the Einstein summation convention is employed. This identity was first derived by Emmy Noether [16]. Equation (1.1) can be written in a completely equivalent but more illuminating form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p_\mu \xi^\mu - H\tau\right] = (\xi^\mu - \dot{q}^\mu\tau)\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu}\right]. \tag{1.2}$$

Extremal or On-shell Motion

The term in square brackets on the right-hand side of Eq. (1.2) is the Euler–Lagrange operator acting on the Lagrangian:

$$E_\mu[L] \equiv \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu}.$$

For a holonomic system, this expression vanishes, so the following conservation law holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[p_\mu \xi^\mu - H\tau\right] = 0. \tag{1.3}$$

For a general non-holonomic system, little can be said. However, if the M constraints are linear in the velocities, so that

$$\gamma^\kappa \equiv A^\kappa_\mu(q,t)\dot{q}^\mu + B^\kappa(q,t) = 0\,, \quad \kappa = 1, \ldots, M,$$

then the d'Alembert–Lagrange equations may be written in the form

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu}\right] = \lambda_\kappa \frac{\partial \gamma^\kappa}{\partial \dot{q}^\mu} = \lambda_\kappa A^\kappa_\mu.$$

The right-hand side of Eq. (1.2) then becomes

$$(\xi^\mu - \dot{q}^\mu\tau)\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^\mu} - \frac{\partial L}{\partial q^\mu}\right] = (\xi^\mu - \dot{q}^\mu\tau)\lambda_\kappa A^\kappa_\mu = \lambda_\kappa(A^\kappa_\mu \xi^\mu + B^\kappa\tau).$$

If we assume that the infinitesimal Noetherian symmetry respects the constraints, namely, if

$$A^\kappa_\mu \xi^\mu + B^\kappa \tau = 0, \qquad \kappa = 1, \ldots, M, \tag{1.4}$$

then this expression vanishes. As a consequence, the right-hand side of Eq. (1.2) vanishes for on-shell flow.

We conclude that, for both holonomic systems and systems subject to non-holonomic constraints that are linear in the velocities, even with inhomogeneous terms, Eq. (1.2) reduces to the conservation law, Eq. (1.3).

2. ROUTH SPHERE

The dynamics of the Routh sphere are discussed in many texts on classical mechanics. The original study is [17]. In this paper we follow the notation of [14] and [15]. There are six degrees of freedom: the configuration of the body is given by (X, Y, Z), the coordinates of the centre of mass, and the three Euler angles (θ, ϕ, ψ). The unit orthogonal triad in the space frame is $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ and the unit orthogonal triad in the intermediate frame is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with $\mathbf{i}$ horizontal and $\mathbf{k}$ fixed along the axis of the body (see Fig. 1).

The holonomic constraint that the geometric centre must remain at unit distance above the underlying plane is used to eliminate the variable Z, leading to an effective system with $N = 5$ degrees of freedom. Assuming unit mass and unit radius, the Lagrangian of the Routh sphere is

$$L = \tfrac{1}{2}\left[(I_1 + a^2 s^2)\dot{\theta}^2 + (I_1 s^2 + I_3 c^2)\dot{\phi}^2 + (2I_3 c)\dot{\phi}\dot{\psi} + (I_3)\dot{\psi}^2 + \dot{X}^2 + \dot{Y}^2\right] - ga(1-c),$$

where $s = \sin\theta$, $c = \cos\theta$ and other notation is conventional. We note that L is independent of both ϕ and ψ. We assume that $I_1 = I_2 \neq I_3$.

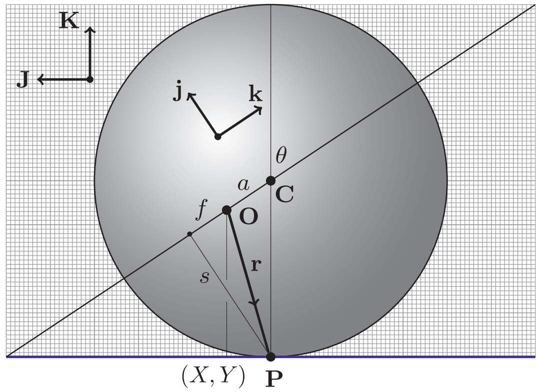

Fig. 1. Geometry and primary coordinates for the Routh sphere. Geometric centre **C**, mass centre **O** and point of contact **P**. In this configuration, **I** and **i** point into the page and $\phi = -\pi/2$.

There are $M = 2$ non-holonomic constraints, which are linear and homogeneous in the velocities, corresponding to rolling motion without slipping:

$$\dot{X} = \quad h s_\phi \dot{\theta} - a s c_\phi \dot{\phi} - s c_\phi \dot{\psi} \tag{2.1}$$

$$\dot{Y} = -h c_\phi \dot{\theta} - a s s_\phi \dot{\phi} - s s_\phi \dot{\psi}, \tag{2.2}$$

where $c_\phi = \cos\phi$, $s_\phi = \sin\phi$ and $h = 1 - ac$ is the height of the centre of mass. We write these constraints in the form $\gamma^\kappa \equiv A^\kappa_\mu \dot{q}^\mu = 0$ where $\dot{q}^\mu = \left(\dot{\theta}, \dot{\phi}, \dot{\psi}, \dot{X}, \dot{Y}\right)$ and

$$A^\kappa_\mu = \begin{bmatrix} -h s_\phi & a s c_\phi & s c_\phi & 1 & 0 \\ h c_\phi & a s s_\phi & s s_\phi & 0 & 1 \end{bmatrix}.$$

For reference, we note that

$$\dot{X}^2 + \dot{Y}^2 = h^2 \dot{\theta}^2 + s^2 (a\dot{\phi} + \dot{\psi})^2.$$

However, we cannot use this to eliminate $\dot{X}$ and $\dot{Y}$ from the Lagrangian as the constraints are non-holonomic [9].

The conjugate momenta are defined in terms of the Lagrangian: $p_\mu = \partial L / \partial \dot{q}^\mu$. For the Routh sphere they are

$$\begin{aligned} p_\theta &= (I_1 + a^2 s^2)\dot{\theta}, \\ p_\phi &= (I_1 s^2 + I_3 c^2)\dot{\phi} + (I_3 c)\dot{\psi}, \\ p_\psi &= (I_3 c)\dot{\phi} + (I_3)\dot{\psi}. \end{aligned}$$

We also have $p_X = \dot{X}$ and $p_Y = \dot{Y}$. Since the determinant of the coefficients (the Hessian) is $(I_1 + a^2 s^2) I_1 I_3 s^2$, we can solve for the velocities:

$$\begin{aligned} \dot{\theta} &= p_\theta / (I_1 + a^2 s^2), \\ \dot{\phi} &= (p_\phi - c p_\psi)/I_1 s^2, \\ \dot{\psi} &= (-c/I_1 s^2) p_\phi + \left((I_1 s^2 + I_3 c^2)/I_1 I_3 s^2\right) p_\psi \end{aligned}$$

and, of course, $\dot{X} = p_X$ and $\dot{Y} = p_Y$.

Invariance

We note that ϕ, ψ, X and Y are all ignorable coordinates. Thus, L is invariant with respect to infinitesimal variations of these coordinates. For free-slip boundary conditions, where there are no constraints linking the momenta, there are four conserved quantities

$$\{p_\phi, p_\psi, p_X, p_Y\}$$

corresponding to these four coordinates.

Since the Lagrangian does not depend explicitly on t, invariance under a transformation of the form $t' = t + \epsilon\tau$ with τ constant leads, in the usual way, to conservation of the energy. We therefore assume a transformation of the space coordinates,

$$\phi' = \phi + \epsilon\,\xi^{\phi}(\theta),$$
$$\psi' = \psi + \epsilon\,\xi^{\psi}(\theta)$$

where the generators are functions of θ, so that

$$\dot\xi^{\phi} = \frac{\mathrm{d}\xi^{\phi}}{\mathrm{d}\theta}\dot\theta \qquad \text{and} \qquad \dot\xi^{\psi} = \frac{\mathrm{d}\xi^{\psi}}{\mathrm{d}\theta}\dot\theta.$$

The constraints also require variations of X and Y of the form

$$X' = X + \epsilon\,\xi^{X}(\theta,\phi),$$
$$Y' = Y + \epsilon\,\xi^{Y}(\theta,\phi),$$

so that ξ^X and ξ^Y depend on ϕ as well as θ. Explicitly, the constraints imply

$$\xi^X = -sc_\phi(a\xi^\phi + \xi^\psi) \qquad \text{and} \qquad \xi^Y = -ss_\phi(a\xi^\phi + \xi^\psi). \tag{2.3}$$

We note that $c_\phi\xi^X + s_\phi\xi^Y = -s(a\xi^\phi + \xi^\psi)$, independent of ϕ. The time derivatives are

$$\dot\xi^X = \big[-c\,c_\phi(a\xi^\phi + \xi^\psi) - sc_\phi(a\xi^\phi_{,\theta} + \xi^\psi_{,\theta})\big]\dot\theta + \big[ss_\phi(a\xi^\phi + \xi^\psi)\big]\dot\phi,$$
$$\dot\xi^Y = \big[-c\,s_\phi(a\xi^\phi + \xi^\psi) - ss_\phi(a\xi^\phi_{,\theta} + \xi^\psi_{,\theta})\big]\dot\theta - \big[sc_\phi(a\xi^\phi + \xi^\psi)\big]\dot\phi.$$

Again, we note that $c_\phi\dot\xi^X + s_\phi\dot\xi^Y$ is independent of ϕ. The invariance identity, Eq. (1.1), now becomes

$$p_\phi\dot\xi^\phi + p_\psi\dot\xi^\psi + p_X\dot\xi^X + p_Y\dot\xi^Y = 0.$$

Substituting the above values we get, for the unconstrained variables,

$$p_\phi\dot\xi^\phi + p_\psi\dot\xi^\psi = [(I_1s^2 + I_3c^2)\xi^\phi_{,\theta} + (I_3c)\xi^\psi_{,\theta}]\dot\theta\dot\phi + [(I_3c)\xi^\phi_{,\theta} + (I_3)\xi^\psi_{,\theta}]\dot\theta\dot\psi$$

and, for the constrained variables,

$$p_X\dot\xi^X + p_Y\dot\xi^Y = \big[s(a\xi^\phi + \xi^\psi) + as^2(a\xi^\phi_{,\theta} + \xi^\psi_{,\theta})\big]\dot\theta\dot\phi + \big[sc(a\xi^\phi + \xi^\psi) + s^2(a\xi^\phi_{,\theta} + \xi^\psi_{,\theta})\big]\dot\theta\dot\psi.$$

Note that this expression is independent of ϕ. Adding these two expressions and setting the coefficients of $\dot\theta\dot\phi$ and $\dot\theta\dot\psi$ separately to zero gives two ODEs for ξ^ϕ and ξ^ψ:

$$(I_1s^2 + I_3c^2 + a^2s^2)\frac{\mathrm{d}\xi^\phi}{\mathrm{d}\theta} + (I_3c + as^2)\frac{\mathrm{d}\xi^\psi}{\mathrm{d}\theta} + s(a\xi^\phi + \xi^\psi) \;=\; 0, \tag{2.4}$$

$$(I_3c + as^2)\frac{\mathrm{d}\xi^\phi}{\mathrm{d}\theta} + (I_3 + s^2)\frac{\mathrm{d}\xi^\psi}{\mathrm{d}\theta} + sc(a\xi^\phi + \xi^\psi) \;=\; 0. \tag{2.5}$$

These are the symmetry equations for the Routh sphere. We can write them

$$\mathsf{F}\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}\theta} = \mathsf{G}\,\boldsymbol{\xi}, \tag{2.6}$$

where $\boldsymbol{\xi} = (\xi^\phi, \xi^\psi)^{\mathrm{T}}$ and the coefficient matrices are

$$\mathsf{F} = \begin{bmatrix} I_1s^2 + I_3c^2 + a^2s^2 & I_3c + as^2 \\ I_3c + as^2 & I_3 + s^2 \end{bmatrix} \qquad \text{and} \qquad \mathsf{G} = -\begin{bmatrix} as & s \\ asc & sc \end{bmatrix}.$$

The determinant of the matrix F is I_1s^2/ρ^2, where

$$\rho = \frac{1}{\sqrt{s^2 + I_3 + (I_3/I_1)f^2}}.$$

So F is invertible and the symmetry equations may be written as $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}\theta = \mathsf{H}\,\boldsymbol{\xi}$, where $\mathsf{H} = \mathsf{F}^{-1}\mathsf{G}$. Explicitly,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix}\xi^{\phi}\\ \xi^{\psi}\end{pmatrix} = \left(-\frac{\rho^2 s}{I_1}\right)\begin{bmatrix} a(I_3+h) & (I_3+h)\\ a(I_1c - I_3c - ha) & (I_1c - I_3c - ha)\end{bmatrix}\begin{pmatrix}\xi^{\phi}\\ \xi^{\psi}\end{pmatrix}. \tag{2.7}$$

Solution of the Symmetry Equations

One solution of Eqs. (2.4) and (2.5) is immediately obvious by inspection: take both ξ^{ϕ} and ξ^{ψ} constant, with $\xi^{\phi} = 1$ and $\xi^{\psi} = -a$. Then $(a\xi^{\phi} + \xi^{\psi}) = 0$ so, by virtue of Eq. (2.3), both ξ^{X} and ξ^{Y} vanish. The Noetherian constant associated with this transformation is

$$C_J = p_\mu \xi^\mu = p_\phi - a p_\psi, \tag{2.8}$$

which is Jellett's constant.

Once a solution of Eqs. (2.7) is known, another one can be found. Suppose there are two linearly independent solutions $(\xi_1^{\phi}, \xi_1^{\psi})^{\mathrm{T}}$ and $(\xi_2^{\phi}, \xi_2^{\psi})^{\mathrm{T}}$. The Wronskian is defined to be the determinant

$$W(\theta) = \begin{vmatrix}\xi_1^{\phi} & \xi_2^{\phi}\\ \xi_1^{\psi} & \xi_2^{\psi}\end{vmatrix} = \xi_1^{\phi}\xi_2^{\psi} - \xi_2^{\phi}\xi_1^{\psi}.$$

It is easily shown that

$$\frac{\mathrm{d}W}{\mathrm{d}\theta} = \mathrm{Tr}\,(\mathsf{H})\,W,$$

where $\mathrm{Tr}\,(\mathsf{H}) = \mathsf{H}_{11} + \mathsf{H}_{22}$. This has a solution $W(\theta) = C\exp[\int \mathrm{Tr}\,(\mathsf{H})\,\mathrm{d}\theta]$. The explicit form of H is implied from Eq. (2.7) so that $\mathrm{Tr}\,(\mathsf{H}) = (-\rho^2 s/I_1)\big[I_1 c - I_3(c-a)\big]$. This can be integrated to yield $W(\theta) = C\rho$, with C a constant depending on the normalisation choice for the linearly independent solutions. Then, using the definition of W, we find that

$$\xi_2^{\phi}(\theta) = \xi_1^{\phi}(\theta)\int^{\theta}\frac{\mathsf{H}_{12}(\theta)}{\xi_1^{\phi}(\theta)^2}W(\theta)\,\mathrm{d}\theta.$$

In the present case, $\xi_1^{\phi}(\theta) = 1$, $\mathsf{H}_{12}(\theta) = (-\rho^2 s/I_1)(I_3 + h)$ and we make the convenient choice $W(\theta) = I_1\rho$. We find, by direct integration, the solution $\xi_2^{\phi}(\theta) = (c-a)\rho$ and thence, since $W = a\xi_2^{\phi} + \xi_2^{\psi}$, we get

$$\begin{pmatrix}\xi_2^{\phi}\\ \xi_2^{\psi}\end{pmatrix} = \begin{pmatrix} f\rho\\ (I_1 - af)\rho\end{pmatrix},$$

where we write $f = c - a$. Equation (2.3) gives ξ^{X} and ξ^{Y}. Then the Noetherian constant is

$$C_R = p_\mu\xi^\mu = \left[\frac{I_1}{I_3}\right]\frac{p_\psi}{\rho}, \tag{2.9}$$

which is Routh's constant.

We can now write the general solution of Eq. (2.7) as

$$\begin{pmatrix}\xi^{\phi}\\ \xi^{\psi}\end{pmatrix} = A_1\begin{pmatrix}\xi_1^{\phi}\\ \xi_1^{\psi}\end{pmatrix} + A_2\begin{pmatrix}\xi_2^{\phi}\\ \xi_2^{\psi}\end{pmatrix} = \begin{pmatrix}A_1 + A_2 f\rho\\ -aA_1 + A_2(I_1 - af)\rho\end{pmatrix}.$$

3. RECOVERING THE SYMMETRY FROM A KNOWN CONSTANT

Suppose we know that $C = p_\mu \xi^\mu$ is a constant of the motion. Then

$$\frac{\partial p_\mu}{\partial p_\nu}\xi^\mu = \left[\frac{\partial p_\phi}{\partial p_\nu}\xi^\phi + \frac{\partial p_\psi}{\partial p_\nu}\xi^\psi + \frac{\partial p_X}{\partial p_\nu}\xi^X + \frac{\partial p_Y}{\partial p_\nu}\xi^Y\right] = \frac{\partial C}{\partial p_\nu} \tag{3.1}$$

provides a system of equations for the generators ξ^μ. For unconstrained motion the momenta are independent and it follows that $\partial p_\mu/\partial p_\nu = \delta^\nu_\mu$, so that

$$\xi^\nu = \frac{\partial C}{\partial p_\nu}.$$

For constrained motion, the generators are interconnected and a linear system of equations must be solved.

We can write the constraint equations (2.1), (2.2) in terms of momenta:

$$\begin{aligned} p_X = &\quad s_\phi \left(\frac{h}{I_1 + a^2 s^2}\right) p_\theta + c_\phi \left[\left(\frac{f}{I_1 s}\right) p_\phi - \left(\frac{fc}{I_1 s} + \frac{s}{I_3}\right) p_\psi\right], \\ p_Y = &-c_\phi \left(\frac{h}{I_1 + a^2 s^2}\right) p_\theta + s_\phi \left[\left(\frac{f}{I_1 s}\right) p_\phi - \left(\frac{fc}{I_1 s} + \frac{s}{I_3}\right) p_\psi\right]. \end{aligned}$$

We also recall that the generators satisfy the constraint equation (2.3):

$$\xi^X = -sc_\phi(a\xi^\phi + \xi^\psi) \qquad \text{and} \qquad \xi^Y = -ss_\phi(a\xi^\phi + \xi^\psi).$$

These expressions allow us to eliminate the momenta p_X and p_Y and the generators ξ^X and ξ^Y from Eq. (3.1) and obtain expressions relating ξ^ϕ and ξ^ψ:

$$\xi^\phi - \left(\frac{f}{I_1}\right)(a\xi^\phi + \xi^\psi) = \frac{\partial C}{\partial p_\phi}, \tag{3.2}$$

$$\xi^\psi + \left(\frac{fc}{I_1} + \frac{s^2}{I_3}\right)(a\xi^\phi + \xi^\psi) = \frac{\partial C}{\partial p_\psi}. \tag{3.3}$$

Let us apply Eqs. (3.2), (3.3) to the Jellett and Routh constants. For the Jellett constant, $C_J = (p_\phi - a p_\psi)$, we have $(\partial C_J/\partial p_\phi, \partial C_J/\partial p_\psi) = (1, -a)$ and the solution is immediately obvious by inspection:

$$\boldsymbol{\Xi}_J \equiv \begin{pmatrix} \xi^\theta \\ \xi^\phi \\ \xi^\psi \\ \xi^X \\ \xi^Y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -a \\ 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

The coordinates X and Y of the centre of mass do not vary. An interpretation of this vector will be given in §4 below.

For the Routh constant, Eq. (2.9), we have $\partial C_R/\partial p_\phi = 0$ and $\partial C_R/\partial p_\psi = I_1/(I_3\rho)$, and Eqs. (3.2), (3.3) become

$$\xi^\phi - \left(\frac{f}{I_1}\right)(a\xi^\phi + \xi^\psi) = 0,$$

$$\xi^\psi + \left(\frac{fc}{I_1} + \frac{s^2}{I_3}\right)(a\xi^\phi + \xi^\psi) = \left[\frac{I_1}{I_3}\right]\frac{1}{\rho}.$$

Eliminating ξ^ψ gives us an expression for ξ^ϕ:

$$\frac{1}{f}\left[I_3 + s^2 + (I_3/I_1)f^2\right]\xi^\phi = \frac{1}{\rho}.$$

Simplifying this we get the infinitesimal Noetherian symmetry

$$\boldsymbol{\Xi}_R \equiv \begin{pmatrix} \xi^\theta \\ \xi^\phi \\ \xi^\psi \\ \xi^X \\ \xi^Y \end{pmatrix} = \rho \begin{pmatrix} 0 \\ f \\ (I_1 - af) \\ -I_1 s c_\phi \\ -I_1 s s_\phi \end{pmatrix}. \tag{3.5}$$

4. INTERPRETATION OF THE ROUTH SPHERE SYMMETRIES

Each infinitesimal Noetherian symmetry associated with a constant of the motion has a geometrical interpretation obtained by integrating it to construct a finite transformation depending on one free parameter. Let us call this free parameter α.

Jellett Symmetry

For the Jellett constant, the Noetherian symmetry (3.4) leads to the equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}\alpha} = 0, \quad \frac{\mathrm{d}\phi}{\mathrm{d}\alpha} = 1, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\alpha} = -a, \quad \frac{\mathrm{d}X}{\mathrm{d}\alpha} = 0, \quad \frac{\mathrm{d}Y}{\mathrm{d}\alpha} = 0.$$

This has the solution

$$\theta = \theta_0, \quad X = X_0, \quad Y = Y_0 \quad \text{(constants)},$$

$$\phi(\alpha) = \alpha + \phi_0, \quad \psi(\alpha) = -a\,\alpha + \psi_0.$$

We consider the virtual motion corresponding to this free parameter α. The angular velocity is simply

$$\boldsymbol{\Omega} = \frac{\mathrm{d}\phi}{\mathrm{d}\alpha}\mathbf{K} + \frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\boldsymbol{k} = \mathbf{K} - a\,\boldsymbol{k} = -\boldsymbol{r},$$

where $\mathbf{K}$ is the unit vector in the vertical direction in the inertial frame, and $\boldsymbol{k}$ is the unit vector in the body frame, pointing along the symmetry axis of the body. The *contact vector* $\boldsymbol{r}$ points from the centre of mass $\boldsymbol{O}$ to the contact point $\boldsymbol{P}$ (see Fig. 1). It follows that $\boldsymbol{\Omega}$ is the vector pointing from the contact point $\boldsymbol{P}$ to the centre of mass $\boldsymbol{O}$.

Since the position of the centre of mass is fixed, while the Euler angle ϕ changes at a constant rate, we deduce that the angular velocity $\boldsymbol{\Omega}$ precesses uniformly about the vertical axis $\mathbf{K}$, describing a cone. The period of this precession is $\Delta\alpha = 2\pi$, the same as the period of the angle ϕ. The period of the angle ψ is $2\pi a$, which is almost never commensurate with 2π. Hence, the motion is generically quasi-periodic.

Routh Symmetry

For the Routh constant, the Noetherian symmetry (3.5) leads to the equations

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\alpha}} = 0, \quad \frac{\mathrm{d}\phi}{\mathrm{d}\tilde{\alpha}} = \rho f, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\tilde{\alpha}} = \rho(I_1 - af), \quad \frac{\mathrm{d}X}{\mathrm{d}\tilde{\alpha}} = -\rho I_1 s\, c_\phi, \quad \frac{\mathrm{d}Y}{\mathrm{d}\tilde{\alpha}} = -\rho I_1 s\, s_\phi. \tag{4.1}$$

Observing that, for θ constant, ρ is a positive constant, we will use the rescaled parameter $\rho\tilde{\alpha}$ as our free parameter α from here on. We can solve the first three equations directly:

$$\theta = \theta_0 \quad \text{(constant)}, \qquad \phi(\alpha) = f\alpha + \phi_0, \quad \psi(\alpha) = (I_1 - af)\alpha + \psi_0, \tag{4.2}$$

where f depends on θ and is thus constant. As in the case of the Jellett symmetry, the angles ϕ and ψ change at constant rates, with ratio $\mathrm{d}\psi/\mathrm{d}\phi = -a + I_1/f$, again incommensurate in general. As θ varies from 0 to π, this ratio may take arbitrary values outside the open interval $(-a - I_1/(1+a), -a + I_1/(1-a))$. In particular, as $I_1 > 0$, it follows that $\mathrm{d}\psi/\mathrm{d}\phi \neq -a$, which shows that the Routh case does not contain the Jellett case.

Let us write the equations for X and Y, the last two equations of (4.1), explicitly, using the partial solutions just found:

$$\frac{\mathrm{d}X}{\mathrm{d}\alpha} = -I_1 s\,\cos(f\alpha + \phi_0), \qquad \frac{\mathrm{d}Y}{\mathrm{d}\alpha} = -I_1 s\,\sin(f\alpha + \phi_0). \tag{4.3}$$

The solution to these is immediate: letting (X_0, Y_0) be the value of (X, Y) at $\alpha = 0$, we have

$$X(\alpha) = -\frac{I_1 s}{f}\left[\sin(f\alpha + \phi_0) - \sin(\phi_0)\right] + X_0, \qquad Y(\alpha) = \frac{I_1 s}{f}\left[\cos(f\alpha + \phi_0) - \cos(\phi_0)\right] + Y_0.$$

The interpretation of this solution is as follows:

- If $f \neq 0$, then the projection of the centre of mass onto the underlying plane describes a circle of radius $R = I_1 s/|f|$, centred at $\big(X_0 + (I_1 s/f)\sin\phi_0, Y_0 - (I_1 s/f)\cos\phi_0\big)$, with period $\Delta\alpha = 2\pi/|f|$. Noting that I_1 and s are non-negative, the sense of rotation of this circular motion is positive if $f > 0$ and negative if $f < 0$. An interesting case is when the parameters I_1, a and the angle θ are such that $I_1 - af = 0$, which requires $f > 0$ in particular. Then the ball does not spin with respect to its symmetry axis: $\psi(\alpha) = \psi_0$ for all α, and thus the motion corresponds to the ball spinning in the positive sense with respect to the vertical axis $\mathbf{K}$: the vector $\boldsymbol{k}$ along the body's symmetry axis precesses about the vertical $\mathbf{K}$ with period $\Delta\alpha$.

- If $f = 0$, namely, if we choose $\theta = \cos^{-1} a$ (which is always possible), then there is no circular motion (the radius tends to infinity): the azimuthal angle ϕ is now constant while the ball spins, and therefore the centre of mass moves on a straight line. The solution of (4.2) and (4.3) in this case is

  $$\phi = \phi_0, \quad \psi(\alpha) = I_1\alpha + \psi_0 \quad X(\alpha) = -I_1 s\alpha\cos(\phi_0) + X_0 \quad Y(\alpha) = -I_1 s\alpha\sin(\phi_0) + Y_0,$$

  so the centre of mass moves in a straight line as the Routh sphere rolls.

5. CHAPLYGIN BALL ON A ROTATING TURNTABLE

The dynamics of a Chaplygin ball on a rotating turntable were analysed in [2] and [18]. The centre of mass of the ball coincides with the geometric centre and $I_1 = I_2 \neq I_3$. The holonomic constraint confines the geometric centre to remain at unit distance above the underlying plane, so that the vertical velocity of the centre of mass vanishes.

Assuming unit mass and unit radius for the Chaplygin ball, the Lagrangian is

$$L = \tfrac{1}{2}\big[I_1\,\dot\theta^2 + (I_1 s^2 + I_3 c^2)\dot\phi^2 + (2I_3 c)\dot\phi\dot\psi + (I_3)\dot\psi^2 + \dot X^2 + \dot Y^2\big],$$

where $s = \sin\theta$, $c = \cos\theta$ as above. The potential energy is constant and is taken to be zero. We note that, as for the Routh sphere, L is independent of both ϕ and ψ.

There are two non-holonomic constraints, which are linear and homogeneous in the velocities, corresponding to rolling motion without slipping with respect to the rotating turntable:

$$\dot X = \quad s_\phi\dot\theta - sc_\phi\dot\psi - \Omega Y, \tag{5.1}$$

$$\dot Y = -c_\phi\dot\theta - ss_\phi\dot\psi + \Omega X, \tag{5.2}$$

where $c_\phi = \cos\phi$ and $s_\phi = \sin\phi$, as above, and Ω is the (constant) angular velocity of the rotating turntable. We write these constraints in the form

$$\gamma^\kappa \equiv A^\kappa_\mu \dot q^\mu + B^\kappa_\mu q^\mu = 0,$$

where $q^\mu = (\theta, \phi, \psi, X, Y)$ and $\dot{q}^\mu = \left(\dot{\theta}, \dot{\phi}, \dot{\psi}, \dot{X}, \dot{Y}\right)$. Thus,

$$A^\kappa_\mu = \begin{bmatrix} -s_\phi & 0 & sc_\phi & 1 & 0 \\ c_\phi & 0 & ss_\phi & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B^\kappa_\mu = \begin{bmatrix} 0 & 0 & 0 & 0 & \Omega \\ 0 & 0 & 0 & -\Omega & 0 \end{bmatrix}.$$

We describe a **systematic method** to find Noetherian symmetries and corresponding constants for the Chaplygin ball on a rotating turntable.

1) We require the symmetries to satisfy the non-holonomic constraints (5.1) and (5.2):
$$-s_\phi \xi^\theta + sc_\phi \xi^\psi + \xi^X + \Omega Y \tau = 0, \tag{5.3}$$
$$c_\phi \xi^\theta + ss_\phi \xi^\psi + \xi^Y - \Omega X \tau = 0. \tag{5.4}$$
Note that the component ξ^ϕ is absent from these equations. The constraints are two linear algebraic equations for the six symmetry components $(\xi^\theta, \xi^\phi, \xi^\psi, \xi^X, \xi^Y, \tau)$, which reduce the number of independent symmetry components to four.

2) We make an *ansatz* for some symmetry components. For example, we might require ξ^ϕ to be the only non-vanishing component.

3) In the invariance identity (1.1), we substitute the symmetry components that are known from the ansatz. This yields a differential equation for the remaining symmetry components.

4) We solve the equation for these components. We can then construct the corresponding conserved quantities, using the invariance identity in the form (1.2).

Symmetry for the Vertical Component of Angular Momentum

Noting that ξ^ϕ does not occur in the non-holonomic constraints (5.3) and (5.4), we seek a symmetry

$$(\xi^\theta, \xi^\phi, \xi^\psi, \xi^X, \xi^Y, \tau) = (0, \xi^\phi, 0, 0, 0, 0).$$

This symmetry automatically satisfies the non-holonomic constraints. Now, because ϕ is an ignorable coordinate, the invariance identity (1.1) becomes $p_\phi \dot{\xi}^\phi = 0$, with solution $\xi^\phi = \text{constant}$. Then the invariance identity in the form (1.2) becomes $\mathrm{d}p_\phi/\mathrm{d}t = 0$, so the ϕ-component of angular momentum

$$L_Z \equiv p_\phi \tag{5.5}$$

is an integral of the motion.

Symmetries for Horizontal Components of Angular Momentum

Noting the unit coefficients of ξ^X and ξ^Y in the constraints (5.3) and (5.4), we seek two types of symmetry:

$$(\xi^\theta, \xi^\phi, \xi^\psi, \xi^X, \xi^Y, \tau) = (\xi^\theta, \xi^\phi, \xi^\psi, 1, 0, 0), \tag{5.6}$$
$$(\xi^\theta, \xi^\phi, \xi^\psi, \xi^X, \xi^Y, \tau) = (\xi^\theta, \xi^\phi, \xi^\psi, 0, 1, 0). \tag{5.7}$$

We consider these symmetries in turn. Substituting (5.6) into the constraints, we easily solve for ξ^θ and ξ^ψ:

$$\xi^\theta = s_\phi, \qquad \xi^\psi = -c_\phi/s. \tag{5.8}$$

These immediately give us expressions for $\dot{\xi}^\theta$ and $\dot{\xi}^\psi$:

$$\dot{\xi}^\theta = c_\phi \dot{\phi}, \qquad \dot{\xi}^\psi = (cc_\phi/s^2)\dot{\theta} + (s_\phi/s)\dot{\phi}.$$

Using these in the invariance identity (1.1), which is $(\partial L/\partial\theta)\xi^\theta + p_\mu\dot{\xi}^\mu = 0$, we obtain an equation for $\dot{\xi}^\phi$:

$$\dot{\xi}^\phi = -\left(\frac{c_\phi}{s^2}\dot{\theta} + \frac{cs_\phi}{s}\dot{\phi}\right).$$

This implies that ξ^ϕ is a function of θ and ϕ only. We obtain

$$\frac{\partial\xi^\phi}{\partial\theta} = -\frac{c_\phi}{s^2}, \qquad \frac{\partial\xi^\phi}{\partial\phi} = -\frac{cs_\phi}{s}.$$

These are easily seen to satisfy the compatibility condition $\partial^2\xi^\phi/\partial\theta\partial\phi = \partial^2\xi^\phi/\partial\phi\partial\theta$ and we immediately have the solution

$$\xi^\phi = \frac{cc_\phi}{s}. \tag{5.9}$$

The final step is to substitute (5.8) and (5.9) into the invariance identity (1.2) to obtain the Noetherian integral

$$L_Y \equiv s_\phi p_\theta + \left(\frac{cc_\phi}{s}\right)p_\phi - \left(\frac{c_\phi}{s}\right)p_\psi + p_X. \tag{5.10}$$

A similar analysis starting from symmetry (5.7) yields the Noetherian integral

$$L_X \equiv c_\phi p_\theta - \left(\frac{cs_\phi}{s}\right)p_\phi + \left(\frac{s_\phi}{s}\right)p_\psi - p_Y. \tag{5.11}$$

Symmetry for an Integral Involving the Energy

To obtain integrals which are non-linear in the velocities, we need to assume $\tau \neq 0$. We seek a symmetry such that

$$(\xi^\theta, \xi^\phi, \xi^\psi, \xi^X, \xi^Y, \tau) = (0, 0, 0, \xi^X, \xi^Y, \tau).$$

The constraints (5.3) and (5.4) then become

$$\xi^X + \Omega Y\,\tau = 0, \tag{5.12}$$

$$\xi^Y - \Omega X\,\tau = 0. \tag{5.13}$$

The invariance identity (1.1) is then

$$p_X\dot{\xi}^X + p_Y\dot{\xi}^Y - H\dot{\tau} = 0.$$

Differentiating the constraints (5.12) and (5.13) and substituting for $\dot{\xi}^X$ and $\dot{\xi}^Y$, we get

$$\big[H - \Omega(X\dot{Y} - \dot{X}Y)\big]\dot{\tau} = 0.$$

This is satisfied for constant τ. Therefore, the invariance identity (1.2) gives us the integral

$$J \equiv H - \Omega L_{\mathrm{O}}, \tag{5.14}$$

where $L_{\mathrm{O}} \equiv (X\dot{Y} - \dot{X}Y) = \mathbf{K}\cdot(\mathbf{R}\times\dot{\mathbf{R}})$ is the the angular momentum due to the motion of the centre of mass about the origin of the space frame ($\mathbf{R} = X\mathbf{I} + Y\mathbf{J}$ is the position vector of the point of contact in the space frame).

Physical Interpretation of the Integrals

The angular momentum about the centre of mass is

$$\mathbf{L}_{\mathrm{C}} \equiv \mathbb{I}_{\mathrm{C}}\boldsymbol{\omega} = I_1\omega_1 + I_2\omega_2 + I_3\omega_3.$$

Following [2], we compute the angular momentum about the point of contact, which is, in our notation,

$$\mathbf{L}_{\mathrm{P}} = \mathbb{I}_{\mathrm{C}}\boldsymbol{\omega} + \mathbf{K}\times(\boldsymbol{\omega}\times\mathbf{K}) - \Omega\mathbf{R}.$$

We note that both the second and third terms on the right are horizontal vectors.

It was shown in [2] that, in the body frame,

$$\left(\frac{\mathrm{d}\mathbf{L}_\mathrm{P}}{\mathrm{d}t}\right)_\mathrm{B} = \mathbf{L}_\mathrm{P} \times \boldsymbol{\omega}. \tag{5.15}$$

Therefore, in the space frame,

$$\left(\frac{\mathrm{d}\mathbf{L}_\mathrm{P}}{\mathrm{d}t}\right)_\mathrm{S} = \left(\frac{\mathrm{d}\mathbf{L}_\mathrm{P}}{\mathrm{d}t}\right)_\mathrm{B} + \boldsymbol{\omega} \times \mathbf{L}_\mathrm{P} = \mathbf{0}.$$

It therefore follows that $F_1 = \mathbf{I}\cdot\mathbf{L}_\mathrm{P}$, $F_2 = \mathbf{J}\cdot\mathbf{L}_\mathrm{P}$ and $F_3 = \mathbf{K}\cdot\mathbf{L}_\mathrm{P}$ are integrals of the motion. Computation of F_3 is simple, since only the first term of (5.15) contributes: $\mathbf{K}\cdot\mathbf{L}_\mathrm{P} = p_\phi$. Expressions for the remaining integrals can be computed:

$$\begin{aligned}\mathbf{I}\cdot\mathbf{L}_\mathrm{P} &= c_\phi p_\theta - \left(\frac{cs_\phi}{s}\right)p_\phi + \left(\frac{s_\phi}{s}\right)p_\psi - p_Y,\\ \mathbf{J}\cdot\mathbf{L}_\mathrm{P} &= s_\phi p_\theta + \left(\frac{cc_\phi}{s}\right)p_\phi - \left(\frac{c_\phi}{s}\right)p_\psi + p_X.\end{aligned}$$

We see that the three components of $\mathbf{L}_\mathrm{P}$ in the space frame are precisely the three integrals (L_X, L_Y, L_Z) that we have derived from Noether's theorem.

In [2], another integral, similar to the Jacobi integral, was found:

$$E = \tfrac{1}{2}\boldsymbol{\omega}\cdot\left[\mathbb{I}_\mathrm{C}\boldsymbol{\omega} + \mathbf{K}\times(\boldsymbol{\omega}\times\mathbf{K})\right] - \tfrac{1}{2}\Omega^2(X^2+Y^2).$$

They cite the origin of this integral as [8]. It is straightforward to show that E is identical to the integral J in (5.14), which we found using Noether's theorem.

Interpretation of the Symmetries

We proceed as in Section 4 to find finite versions of the four infinitesimal symmetries just found.

- **Symmetry for L_Z.** In terms of the free parameter α of the symmetry, we get the equation

$$\frac{d\phi}{d\alpha} = 1,$$

with solution $\phi(\alpha) = \alpha$. The remaining coordinates (θ, ψ, X, Y) are kept constant. This corresponds geometrically to the spinning of the ball about the point of contact, at a constant angular velocity $d\phi/d\alpha = 1$.

- **Symmetry for L_Y.** Reading off the coefficients of p_μ from equation (5.10), we get the equations

$$\frac{d\theta}{d\alpha} = \sin\phi\,, \qquad \frac{d\phi}{d\alpha} = \frac{\cos\theta\cos\phi}{\sin\theta}, \qquad \frac{d\psi}{d\alpha} = -\frac{\cos\phi}{\sin\theta}, \qquad \frac{dX}{d\alpha} = 1, \qquad \frac{dY}{d\alpha} = 0.$$

One immediately gets $X(\alpha) = \alpha$ and $Y =$ constant. This suggests that the geometric interpretation of this symmetry corresponds to a rotation of the ball such that Y is constant and X changes linearly. To see this, consider the equations for θ and ϕ. They provide the first integral

$$\sin\theta\cos\phi = y_0 \quad \text{(constant)},$$

which validates this interpretation. A less obvious result follows from the equation for ψ, which can be solved by quadrature, giving the implicit first integral

$$\tan(\psi-\psi_0) = \cos\theta\cot\phi \qquad\qquad (\psi_0 = \text{constant}).$$

- **Symmetry for L_X.** Reading off the coefficients of p_μ from Eq. (5.11), we get the equations

$$\frac{d\theta}{d\alpha} = \cos\phi, \qquad \frac{d\phi}{d\alpha} = -\frac{\cos\theta\sin\phi}{\sin\theta}, \qquad \frac{d\psi}{d\alpha} = \frac{\sin\phi}{\sin\theta}, \qquad \frac{dX}{d\alpha} = 0, \qquad \frac{dY}{d\alpha} = -1.$$

Here the interpretation of the symmetry corresponds to a rotation of the ball such that X is constant and Y changes linearly. In a similar fashion to the results obtained for L_Y, we obtain the following first integrals:

$$\sin\theta\sin\phi = x_0 \quad \text{(constant)}, \qquad\qquad \tan(\psi-\psi_0) = -\cos\theta\tan\phi \qquad\qquad (\psi_0 = \text{constant}).$$

6. DISCUSSION

The key property of the infinitesimal Noetherian symmetries found for the Routh sphere and the Chaplygin ball is that they satisfy the non-holonomic constraints. In the more general case of the Chaplygin top or the Rock'n'roller, it is not known whether an infinitesimal Noetherian symmetry that satisfies the non-holonomic constraints exists.

If such a symmetry existed, then a constant of motion could be constructed via Eq. (1.3). For example, it is possible to show for these more general cases that the transformation

$$\phi \to \phi + \epsilon$$

(while keeping all other variables unchanged, including X and Y) is an infinitesimal Noetherian symmetry. However, this symmetry does not satisfy the non-holonomic constraints (2.1), (2.2) (with velocities replaced by the generators). In fact, from Eq. (1.2) we obtain

$$\frac{\mathrm{d}p_\phi}{\mathrm{d}t} = as(\lambda_1 c_\phi + \lambda_2 s_\phi),$$

where λ_1 and λ_2 are the multipliers associated with the constraints (2.1) and (2.2), respectively. This example shows that a Noetherian symmetry is potentially useful even if it does not satisfy the non-holonomic constraints: it provides direct formulas for the total time derivative of quantities, which in principle could be exploited for applications such as finding Lyapunov functions.

Another avenue of research is the understanding of the Lie algebra between the Noetherian symmetries that we found for non-holonomic systems. In the case of holonomic systems, it is well known that the Lie bracket between two symmetries is another symmetry. This leads to a method for finding new integrals starting from known ones [5]. However, when non-holonomic constraints are imposed, the usual Lie bracket between two Noetherian symmetries does not necessarily produce another Noetherian symmetry. Further research on the relation between Poisson brackets and symmetries (see [3, 7] for studies in the context of the Routh sphere) is needed to generalise the Lie bracket as a method to produce new Noetherian symmetries.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. Bahar, L. Y. and Kwanty, H. G., Extension of Noether's Theorem to Constrained Non-Conservative Dynamical Systems, *Int. J. Nonlin. Mech.,* 1987, vol. 22, no. 2, pp. 125–138.
2. Bizyaev, I. A., Borisov, A. V., and Mamaev, I. S., Dynamics of the Chaplygin Ball on a Rotating Plane, *Russ. J. Math. Phys.,* 2018, vol. 25, no. 4, pp. 423–433.
3. Bizayev, I. A. and Tsiganov, A. V., On the Routh Sphere Problem, *J. Phys. A,* 2013, vol. 46, 085202, 11 pp.
4. Borisov, A. V. and Mamaev, I. S., Symmetries and Reduction in Nonholonomic Mechanics, *Regul. Chaotic Dyn.,* 2015, vol. 20, no. 5, pp. 553–604.
5. Bustamante, M. D. and Hojman, S. A., Lagrangian Structures, Integrability and Chaos for 3D Dynamical Equations, *J. Phys. A,* 2003, vol. 36, no. 1, pp. 151–160.
6. Chaplygin, S. A., On a Ball's Rolling on a Horizontal Plane, *Regul. Chaotic Dyn.,* 2002, vol. 7, no. 2, pp. 131–148; see also: *Math. Sb.,* 1903, vol. 24, no. 1, pp. 139–168.
7. Cushman, R., Routh's Sphere, *Rep. Math. Phys.,* 1998, vol. 42, nos. 1–2, pp. 47–70.
8. Fassò, F., García-Naranjo, L. C., and Sansonetto, N., Moving Energies As First Integrals of Nonholonomic Systems with Affine Constraints, *Nonlinearity,* 2018, vol. 31, no. 3, pp. 755–782.
9. Flannery, M. R., The Enigma of Nonholonomic Constraints, *Am. J. Phys.,* 2005, vol. 73, no. 3, pp. 265–272.
10. Gray, C. G. and Nickel, B. G., Constants of the Motion for Nonslipping Tippe Tops and Other Tops with Round Pegs, *Am. J. Phys.,* 2000, vol. 68, no. 9, pp. 821–828.

11. Kim, B., Routh Symmetry in the Chaplygin's Rolling Ball, *Regul. Chaotic Dyn.,* 2011, vol. 16, no. 6, pp. 663–670.
12. Kozlov, V. V., On the Integration Theory of Equations of Nonholonomic Mechanics, *Regul. Chaotic Dyn.,* 2002, vol. 7, no. 2, pp. 161–176.
13. Kostmann-Schwarzbach, Y., *The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century,* Boston: Springer, 2011.
14. Lynch, P. and Bustamante, M. D., Precession and Recession of the Rock'n'Roller, *J. Phys. A,* 2009, vol. 42, no. 42, 425203, 25 pp.
15. Lynch, P. and Bustamante, M. D., Quaternion Solution for the Rock'n'Roller: Box Orbits, Loop Orbits and Recession, *Regul. Chaotic Dyn.,* 2013, vol. 18, nos. 1–2, pp. 166–183.
16. Noether, E., Invariente Variationsprobleme, *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. II,* 1918, vol. 1918, pp. 235–257.
17. Routh, E. J., *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies: Being Part II of a Treatise on the Whole Subject,* 6th ed., New York: Dover, 1955.
18. Tsiganov, A. V., Hamiltonization and Separation of Variables for a Chaplygin Ball on a Rotating Plane, *Regul. Chaotic Dyn.,* 2019, vol. 24, no. 2, pp. 171–186.