The Fractal Boundary of the Power Tower Function[1]

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Abstract

We consider the function called the *power tower function*, defined by iterated exponentiation (or tetration) of the complex variable z. For real values x, it converges on the interval $\exp(-e) < x < \exp(1/e)$. The function may be expressed as the inverse of the function $x = y^{1/y}$, allowing an extension of the domain to $0 < x < \exp(1/e)$. It may also be expressed in terms of the Lambert W-function, enabling an analytical continuation to the complex plane.

The boundary of the region of the complex plane for which the power tower converges to a finite value is fractal in nature. We show this by repeatedly zooming to higher magnifications, illustrating the deliciously intricate nature of the boundary.

1 The Power Tower Function for Real Argument

We will consider the function of a real variable x, defined by

$$y(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \tag{1}$$

It would seem that when $x > 1$ this must blow up. Amazingly, this is not so. In fact, Euler showed that the function converges for values $\exp(-e) < x < \exp(1/e)$ or approximately $0.066 < x < 1.445$. We call this function the *power tower function.* (Repeated exponentiation is called *tetration* and the function is sometimes called the *tetration function.*)

Let us consider the sequence of approximations

$$\{y_1, y_2, y_3, \dots\} = \left\{x, x^x, x^{x^x}, \dots\right\}$$

We note the convention

$$x^{x^x} \equiv x^{(x^x)} \qquad \text{and } \textit{not} \qquad x^{x^x} = (x^x)^x = x^{x^2}\,.$$

Thus, the tower is constructed *downwards.* It should really be denoted as

$$y(x) = {}^{\cdot^{\cdot^{\cdot}}} x^{x^x}$$

as each new x is adjoined to the *bottom* of the tower.

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Behaviour for Large and Small x

We evaluate the sequence

$$\left\{x, x^x, x^{x^x}, \dots\right\}$$

for several particular values of x. We find that we may get (a) convergence to a finite value, (b) divergence to infinity, (c) oscillation between two or more values, or (d) irregular (chaotic) behaviour.

For $x = 1$, every term in the sequence is equal to 1 so the limiting value is 1. The choice $x = \sqrt{2}$ is interesting: the values grow towards $y = 2$. This is confirmed by substitution in (2) below. For $x = 1.5$, the tower grows without limit. For $x < 1$ we see oscillating behaviour, converging if x is not too small. For very small values, there is oscillation but *no convergence.*

It is clear that $\lim_{x\to\infty} x^x = \infty$, so for large x the power tower function diverges. What is less obvious is that $\lim_{x\to 0} x^x = 1$. For small x, alternate terms are close to 0 and to 1. This accounts for the oscillating but non-converging behaviour of the sequence for very small x.

Iterative Process

If the power tower function is to have any meaning, we need to show that it has well-defined values. We consider the iterative process with $y_1 = x$ and

$$y_{n+1} = x^{y_n} .$$

This generates the infinite sequence

$$\{y_1, y_2, y_3, \dots\} = \left\{x, x^x, x^{x^x}, \dots\right\}$$

If the sequence converges to $y = y(x)$, it follows that

$$y = x^y . \tag{2}$$

But this leads to an explicit expression for x:

$$x = y^{1/y} \tag{3}$$

Taking the derivative of this function we get

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{1 - \log y}{y^2}\right) x$$

which vanishes when $\log y = 1$ or $y = e$. At this point, $x = \exp(1/e)$. Moreover, it is easily shown that

$$\lim_{y\to 0} x = 0 \qquad \text{and} \qquad \lim_{y\to\infty} x = 1$$

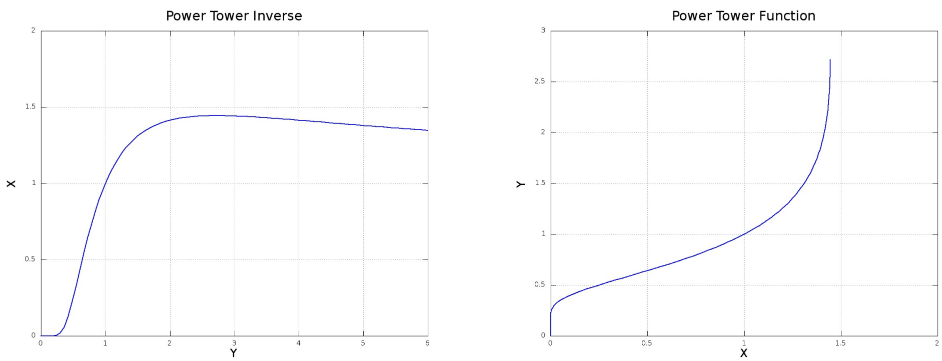

Figure 1: Left: $x = y^{1/y}$ for $y \in (0, 6)$. Right: Power tower function for $0 < x < \exp(1/e)$.

The function $x = y^{1/y}$ is plotted in Fig. 1. It is defined for all positive y. Its derivative vanishes at $y = e$ where it takes its maximum value $\exp(1/e)$. It is monotone increasing on the interval $(0, e)$ and has an inverse function on this interval. This inverse is the power tower function (Fig. 1, right panel).

The logarithm of $y = x^y$ gives $\log y = y \log x$. That is

$$y = \exp(y \log x) \qquad \text{or} \qquad y = \exp(\xi y) \tag{4}$$

where $\xi = \log x$. This is suited for iterative solution: given a value of x (or ξ), we seek a value y such that the graph of $\exp(\xi y)$ intersects the diagonal line $y = y$. Starting from some value $y_{(0)}$, we iterate:

$$y_{(n+1)} = \exp(\xi y_{(n)})$$

(see, e.g., [4], p. 315). In Fig. 2, we sketch the graph of $\exp(\xi y)$ for a selection of values of ξ. For $\xi < 0$, corresponding to $x < 1$, there is a single root of (4) (Fig. 2, top left panel). For $0 < \xi < 1/e$ (that is, for $1 < x < e^{1/e}$), there are two roots (top right panel). For $\xi = 1/e$ ($x = e^{1/e}$), there is one double root (bottom left panel). Finally, for $\xi > 1/e$ ($x > e^{1/e}$), there are no roots (bottom right panel).

We compute iterations of (4). The iterative method converges only if the derivative

$$\frac{\mathrm{d}}{\mathrm{d}y} \exp(\xi y) = \xi y$$

of the right side has modulus less than unity. This criterion is satisfied for $-e < \xi < 0$, and also for the smaller of the two roots when $0 < \xi < 1/e$. We therefore expect to obtain a single solution for $-e < \xi < 1/e$ or $\exp(-e) < x < \exp(1/e)$.

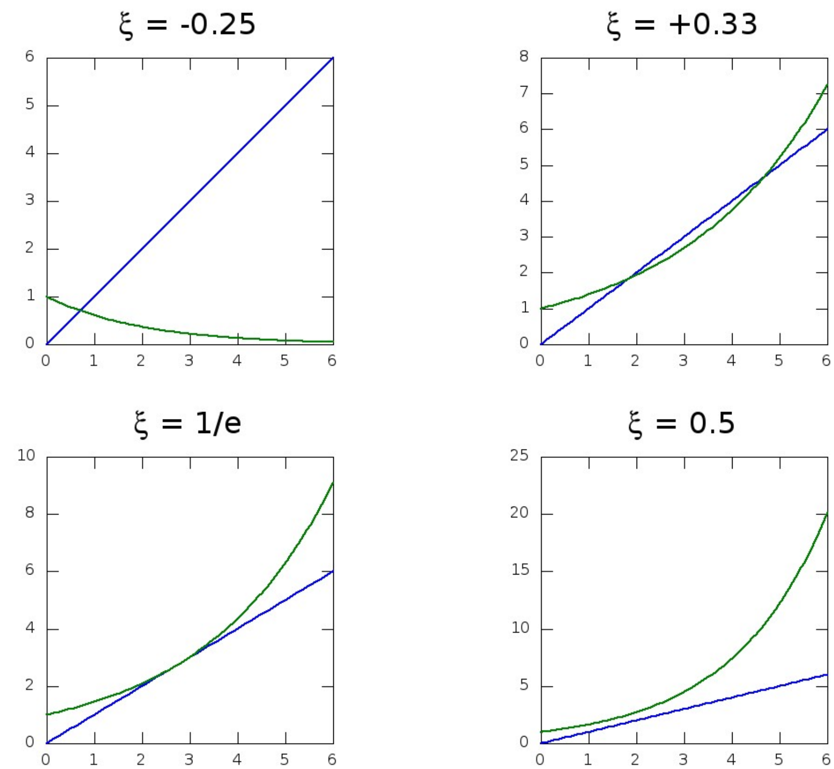

Figure 2: Graphs of $y = y$ and $y = \exp(\xi y)$ for some values ξ

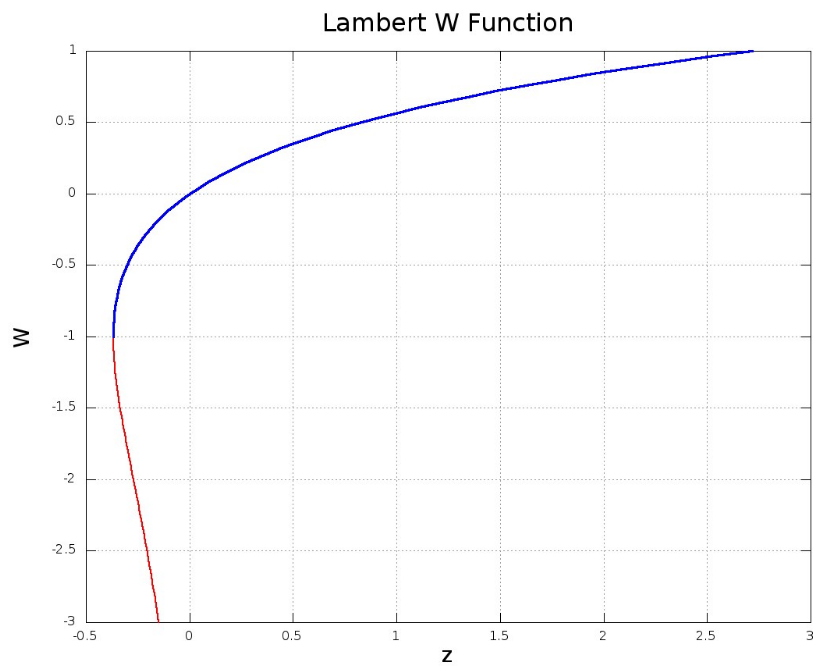

Figure 3: Lambert W-function $w = W(z)$. The inverse of $z = w \exp(w)$.

2 The Lambert W-Function

Swiss mathematician *Johann Heinrich Lambert* (1728–1777) introduced a function that is of wide value and importance. The Lambert W-function is the inverse of $z = w \exp(w)$:

$$w = W(z) \qquad \Longleftrightarrow \qquad z = w \exp(w)\,.$$

A plot of $w = W(z)$ is presented in Fig. (3). We confine attention to real values of $W(z)$, which means that $z \geq -1/e$. The W-function is single-valued for $z \geq 0$ and double-valued for $-1/e < z < 0$. The constraint $W(z) > -1$ defines a single-valued function on $z \in [-1/e, +\infty)$. This is the principal branch, denoted when appropriate as $W_0(z)$. The other branch, real on $z \in [-1/e, 0)$, is denoted $W_{-1}(z)$. For further details, see [1].

The Lambert W-function occurs frequently in mathematics and physics (see Table 1). Indeed, it has been "re-discovered' several times in various contexts. In pure mathematics, the W-function is valuable in solving transcendental and differential equations, in combinatorics (as the Tree function), for delay differential equations and for iterated epxonentials (which is the context in which we have introduced it). In theoretical computer science, it is used in the analysis of algorithms. Physical applications include water waves, combustion problems, population growth, eigenstates of the hydrogen molecule and, recently, quantum gravity.

The W-function also serves as a pedagogical aid. It is a useful example in introducing implicit functions. It is also a valuable test case for numerical solution methods. In the

Table 1: Applications of the W-Function

MATHEMATICS	PHYSICS
Transcendental equations.	Analysis of algorithms.
Solving differential equations.	Water waves.
In combinatorics.	Combustion problems.
Delay differential equations.	Population growth.
Iterated exponentials.	Eigenstates of H_2 molecule.
Asymptotics.	Quantum gravity.

context of complex variable theory, it is a simple example of a function with both algebraic and logarithmic singularities. Finally, it has a range of interesting asymptotic behaviours. For further references on this, see [1].

Power Tower Function and W

For the Power Tower Function, x in terms of y is:

$$x = y^{1/y}$$

This is well defined for all positive y. Its inverse has a branch point at $(x, y) = (e^{1/e}, e)$. If $\xi = \log x$ we have $y = \exp(\xi y)$. We can write

$$(-\xi y)\exp(-\xi y) = (-\xi)$$

We now define $z = -\xi$ and $w = -\xi y$ and have $z = w\exp(w)$. By the definition of the Lambert W-function, this is

$$w = W(z)$$

Returning to variables x and y, we conclude that

$$y = \frac{W(-\log x)}{-\log x} \tag{5}$$

which is the expression for the power tower function in terms of the Lambert W-function.

The relationship between the power tower function and the Lambert W-function allows us to use *analytical continuation* to extend the power tower function to the complex plane. The function has a logarithmic branch point at $x = 0$. The behaviour of the different branches of the W-function are described in [1].

3 The Imaginary Power Tower

We now examine the power tower function for complex z. Specifically, we look at the case $z = i$:

$$q = i^{i^{i^{\cdot^{\cdot^{\cdot}}}}} .$$

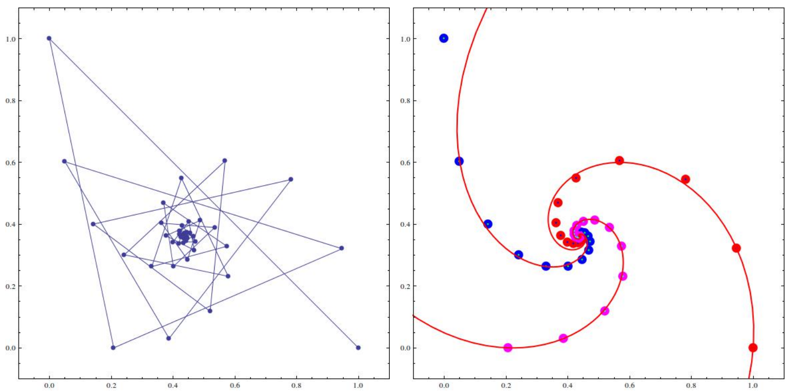

Figure 4: Iteration sequence for $z = i$. Left: the points joined by straight line segments. Right: Sequence points with three logarithmic spirals superimposed.

The first few terms of the sequence are

$$q_1 = i \qquad q_2 = i^i = e^{-\pi/2} \qquad \dots \qquad q_{n+1} = i^{q_n} \qquad \dots$$

Assuming the sequence $\{q_n\}$ converges to Q, we have

$$Q = i^Q$$

Writing $Q = \varrho \exp(i\vartheta)$, it follows that

$$\vartheta \tan \vartheta = \log \left[\frac{\pi}{2} \frac{\cos \vartheta}{\vartheta} \right] \qquad \text{and} \qquad \varrho = \frac{2}{\pi} \frac{\vartheta}{\cos \vartheta}$$

These equations are easily solved numerically to give

$$Q = (0.438283, 0.360592)$$

In Fig. 4 we show the sequence $\{q_n\}$. The points spiral around the limit point Q, converging towards it. The points q_n fall into three distinct sets. Three logarithmic spirals are superimposed on the plot (Fig. 4, right panel). Is this pattern accidental?

We fitted a single logarithmic spiral to the sequence $\{z_n(i)\}$. The points of the sequence were close to such a curve but did not lie exactly upon it. Therefore, we looked at the *asymptotic behaviour* of the sequence for large n.

We consider the specific case $z = i$ and suppose that $z_n = (1 + \epsilon)Z$ where ϵ is small. Then we find that $z_{n+1} = Z^\epsilon \cdot Z$ so that

$$\left(\frac{z_{n+1} - Z}{z_n - Z} \right) = \left(\frac{Z^\epsilon - 1}{\epsilon} \right) .$$

By L'Hôpital's rule, the limit of the right-hand side as $\epsilon \to 0$ is $\log Z$. Thus for small ϵ (large n) we have

$$(z_{n+1} - Z) \approx \log Z \cdot (z_n - Z)$$

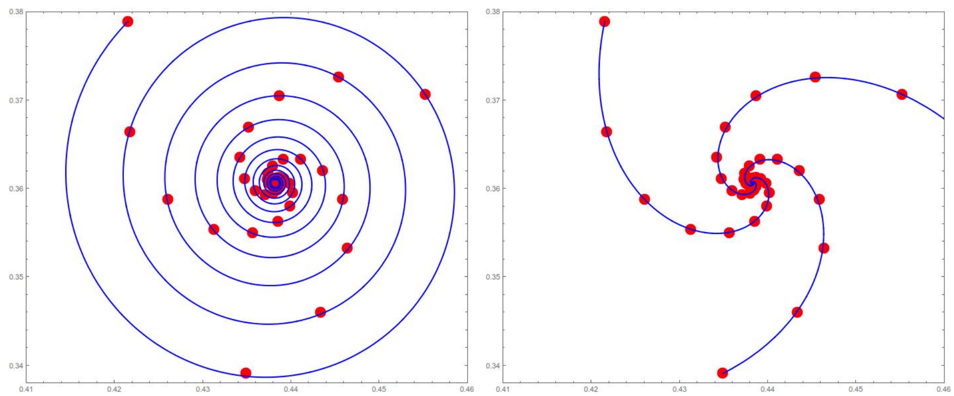

Figure 5: Left: Logarithmic spiral superimposed on the points $\{z_n(i)\}$ for $n \geq 30$. Right: Same sequence of points with three supernumerary spirals superimposed.

and the sequence of differences $\{z_{n+k} - Z\}$ lies approximately on a logarithmic spiral

$$z_{n+k} \approx Z + (\log Z)^k \cdot (z_n - Z) \, .$$

The points $\{z_n(i)\}$ for $n \geq 30$ spiral around the limit point $(0.438283, 0.360592)$ The logarithmic spiral (Fig. 5) gives an excellent fit.

The points z_n fall into three sets. In Fig. 5 (right panel) three logarithmic spirals are superimposed. These "supernumerary spirals" are no accident. Such spirals are familiar in many contexts: In the seeds of a sun-flower, clockwise and anti-clockwise spirals are evident. By changing the parameter z it is possible to tune the limit $Z(z)$ to have spirals of a particular shape. Patterns like this also found in pursuit problems.

4 Power Tower Fractal

We can construct a beautiful fractal set using the power tower function with complex arguments. We examine the behaviour of the (tetration) function

$${}^{\infty}z = z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}$$

For some values of z this *converges*. For other values it is *periodic*. For others, it *"escapes" to infinity*. The boundary of the region for which the function is finite is fractal.

Let $\mathbf{\Pi}$ be the set for which ${}^{\infty}z$ is finite. The "escape set" is the complement of this set. The boundary of the set $\mathbf{\Pi}$ is exquisitely complex. Fig. 6 shows the escape region in light yellow and the region where the sequence is bounded in dark blue. Repeated zooms show how new details continue to emerge at each scale.

The images of the fractal were generated using code acquired from the website of Peter Young [8] for computing the Mandelbrot fractal. It was a simple matter to replace the equation

($z_{n+1} = z_n^2 + c$) defining the Mandelbrot set by the equation ($z_{n+1} = z^{z_n}$) defining the power tower sequence. Zooming in can be continued indefinitely, revealing ever more sturcture. Fig. 7 shows more details of the fractal in a region corresponding to the bottom right-hand panel of Fig. 6.

The fine details at any resolution are not reliable. Structures that appear to be disjoint may be connected by fine filaments that are visible only at higher resolution. It is necessary to set the escape radius to a very large value (e.g. $r_{max} = 10^{48}$) and allow many iterations. The power tower fractal appears to hold riches comparable to those of the much-studied Mandelbrot set.

Much more may be said about the power tower fractal. We could consider fixed points, for which ${}^{\infty}z = z$. Clearly, $z = 1$ and $z = -1$ are fixed points. There are periodic orbits (see `http://www.tetration.org/`). *Sarkovskii's Theorem* implies that a map containing period three must contain all periods from one to infinity. Many other interesting questions remain to be answered.

5 Summary

The power tower function (1) is well defined for real arguments on the domain $(\exp(-e), \exp(1/e))$. Convergence for $x > 1$ is surprising, as a cursory glance would suggest divergence such values. The value of the function may be computed directly from (1) on $[\exp(-e), \exp(1/e)]$ or as the inverse of the function $x = y^{1/y}$ for $y \in (0, e)$. The function $x = y^{1/y}$ tends to zero faster than any polynomial function as $y \to 0$. Correspondingly, the power tower function is 'steeper' than any inverse root of x as $x \to 0$.

The relationship (5) between the power tower function and the Lambert W-function allows us to extend the definition of the power tower function to the complex plain. The function has a logarithmic branch point at $x = 0$. The behaviour of the different branches of the W-function are described in [1].

The boundary of the region of convergence of the power tower function is deliciously complex. It is fractal in nature and successive zooms reveal ever more details. Computation is more difficult than for the Mandelbrot fractal, as that set is bounded whereas the power tower fractal is not. Much remains to be learned about this fascinating mathematical structure.

Appendix: Iteration for $\mathbf{x < e^{-e}}$

The iterative sequence $\{y_n(x)\}$ converges for $\exp(-e) < x < \exp(1/e)$. For $0 < x < \exp(-e)$, it does not converge, but is found to alternate between two values. Denoting these by a and b, we must have

$$x^a = b \qquad \text{and} \qquad x^b = a$$

This leads to $a^a = b^b$, but we cannot conclude that $a = b$, because the function $y = x^x$ has a turning point at $x = 1/e$ (see Fig. 8, left panel).

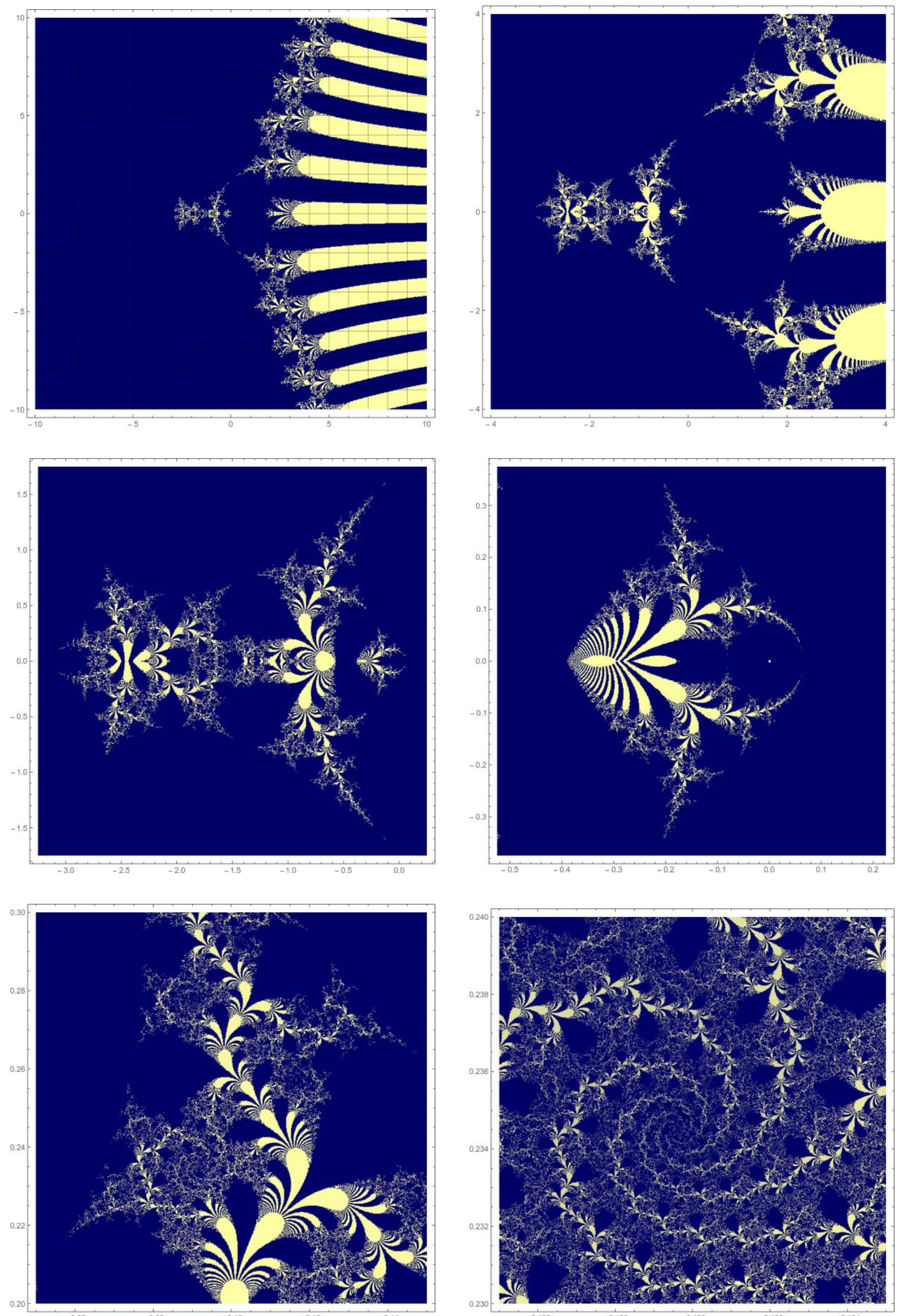

Figure 6: The power tower fractal. Top left: $|x| < 10, |y| < 10$. Top right: $|x| < 4, |y| < 4$. Middle left: $-3.25 \leq x \leq 0.25$, $-1.75 \leq y \leq 1.75$. Middle right: $-0.525 \leq x \leq 0.225$, $-0.375 \leq y \leq 0.375$. Bottom left: $-0.23 \leq x \leq -0.13$, $+0.2 \leq y \leq 0.3$. Bottom right: $-0.193 \leq x \leq -0.183$, $+0.23 \leq y \leq 0.24$.

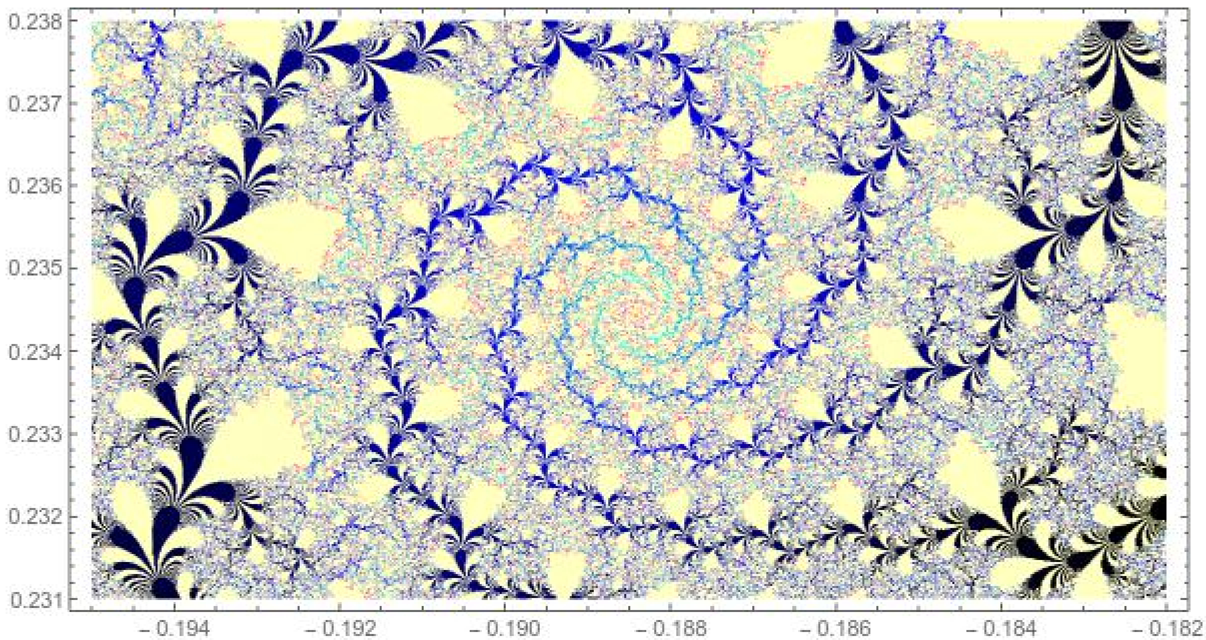

Figure 7: Detail of the power tower fractal in the vicinity of a spiral structure. The region is that of the bottom right-hand panel of Fig. 6.

In Fig. 8 (right panel) we show the solution of the iterative procedure (4) for $0 < x < 0.1$. When $x > \exp(-e)$, there is convergence. When $x < \exp(-e)$, the sequence alternates between two values, shown by the upper and lower curves in the figure. The central curve for $0 < x < \exp(-e)$ is the inverse of $x = y^{1/y}$. There is a pitchfork bifurcation at $x = \exp(-e)$.

The failure of the sequence $\{y_n(x)\}$ to converge for $x < \exp(-e)$ is explained in terms of the slope of the exponential function. The iterative approximation to $y = \exp(\xi y)$ diverges if the absolute value of the gradient of the exponential exceeds 1. But the gradient is $\xi \exp(\xi y) = \xi y$ and this is -1 when $y = 1/e$ and $x = e^{-e}$. For smaller x, the process diverges.

Although the sequence $\{y_n(x)\}$ does not converge for $0 < x < \exp(-e)$, the power tower function may be defined on this interval by the process of analytic continuation. The function $x = y^{1/y}$ has a unique real value on the interval $y \in (0, e)$, so the power tower function may be defined as the inverse of this function, with domain $0 < x < \exp(1/e)$ and range $y \in (0, e)$.

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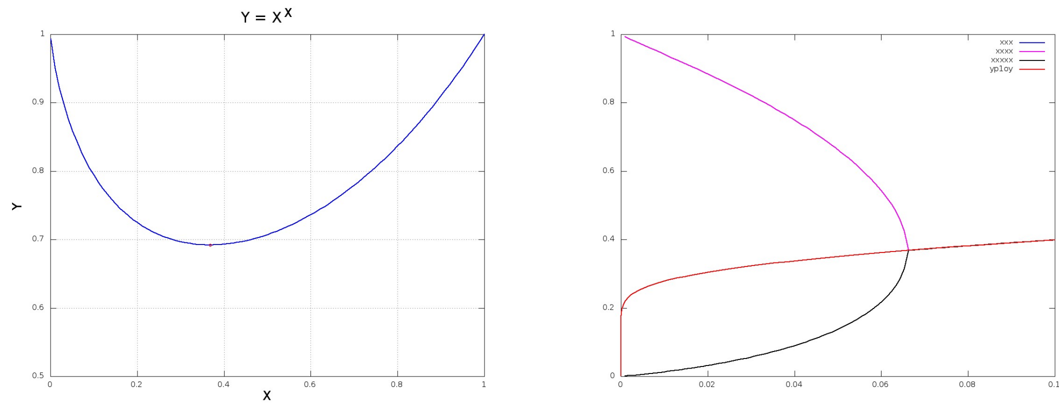

Figure 8: Left: Graph of $y = x^x$ for $0 < x < 1$. Right: Solution of the iterative procedure (4) for $0 < x < 0.1$, showing a pitchfork bifurcation at $x = \exp(-e)$.

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