In Retrospect

Replication of Foucault's pendulum experiment in Dublin

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Abstract

Léon Foucault's pendulum experiment in 1851 generated widespread interest. The experiment was repeated in numerous locations in Europe and the United States of America. The more careful of these demonstrations confirmed the effect of the Earth's rotation on the precession of the swing-plane of the pendulum. A set of pendulum experiments were carried out by Joseph Galbraith and Samuel Haughton in Dublin and a comprehensive mathematical analysis of them was published in 1851.

'You are invited to see the Earth turn...'

A large crowd of spectators gathered in the Panthéon in Paris in March 1851 to witness evidence of the Earth spinning before their eyes. With a simple apparatus comprising a heavy bob swinging on a wire, Léon Foucault showed how the Earth rotates on its axis. His pendulum demonstration caused a sensation, and Foucault achieved instant and lasting fame. The pendulum was 67 metres in length with a bob of 28kg in mass. It swung through a diameter of about six metres so the amplitude (swing-angle) was about 5°. The position of the bob was indicated on a large circular scale marked in quarters of a degree (Fig. 1).

The announcement of the experiment read 'You are invited to see the Earth turn...'. Demonstrations were held daily, and attracted large crowds. The experiment generated widespread international interest and by June 1851 it had been repeated in many cities in Europe and the United States of America, including Dublin.

Several previous attempts to illustrate the Earth's rotation by mechanical means had been made, but with limited success. Both Pierre-Simon Laplace and Carl Friedrich Gauss had been involved in the analysis of deflections from the vertical of objects dropped down mineshafts, but the

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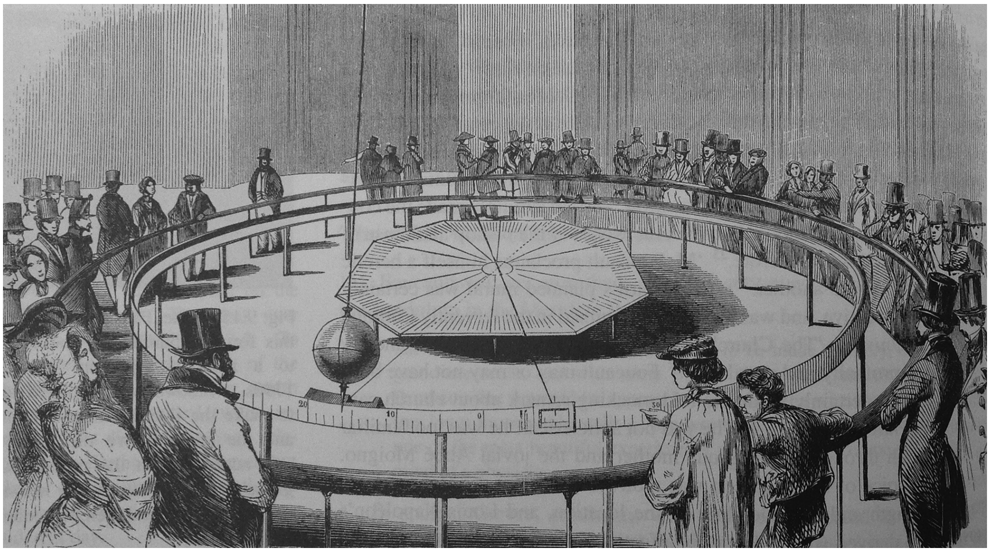

FIG. 1—Engraving in *L'Illustration* of Foucault's pendulum in the Panthéon, Paris.

effect of rotation was too small to be detected with confidence. Foucault's demonstration changed matters. The tiny deflections evident during each swing of the pendulum accumulate with time so that a large veer (or clockwise turning) is observed over a period of an hour or so. This is why Foucault's demonstration succeeded where earlier attempts to illustrate the Earth's rotation failed.

In the immediate aftermath of Foucault's demonstration, pendulum mania raged across Europe and the United States and the experiment was repeated hundreds of times.[1] Many of these attempts were undertaken without due care. The *London Literary Gazette* reported on several cases in which to the horror of the spectators, the Earth had been shown to turn the wrong way. These errors were probably due to a lack of care in setting up the experiments, elliptical bob trajectories resulting from incorrect starting conditions or stray air currents.

Galbraith and Haughton

In September 1851 the *American Journal of Science* surveyed several pendulum demonstrations in Europe and the United States.[2] This review included brief details of the experiments carried out in Dublin by two members of the Royal Irish Academy.

Joseph Galbraith (1818–90) and Samuel Haughton (1821–97) were close contemporaries and lifelong friends (Fig. 2). Both men were highly talented applied mathematicians. Both took holy orders within the Church of Ireland. Both were members of the Royal Irish Academy and both were

[1] Michael F. Conlin, 'The popular and scientific reception of the Foucault pendulum in the United States', *Isis* 90:2 (1999), 181–204.
[2] C. S. Lyman, 'Observations on the pendulum experiment', *American Journal of Science* 12 (1851), 251–5.

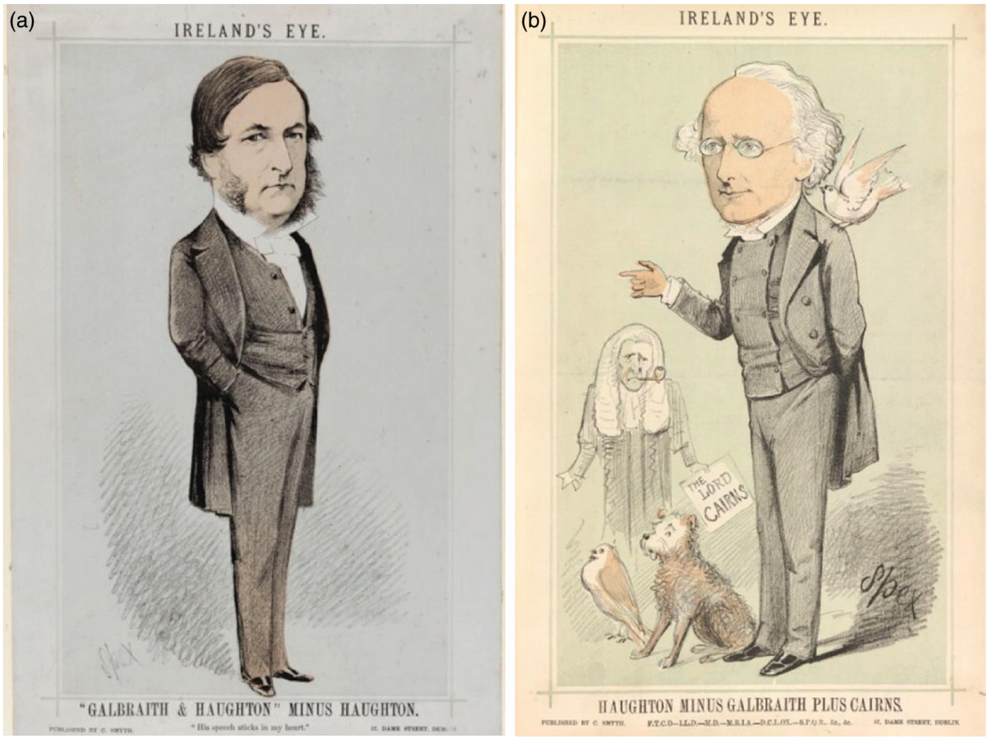

FIG. 2—Cartoons from *Ireland's Eye*, October 1874. (a) '"Galbraith & Haughton" minus Haughton'. (b) 'Haughton minus Galbraith plus Cairns'. Courtesy of the Board of Trinity College Dublin Library.

Freemasons in the same lodge. Galbraith's diary (see Appendix A) shows that the two men interacted frequently, often walking or dining together, and their close friendship was sealed by the marriage of a son of Haughton to a daughter of Galbraith. Both men sympathised with Irish nationalism, Galbraith being active in the Home Rule movement. There is no full-length biography of either scholar, but a good account of their parallel lives is given in an article by Miguel De Arce (2011).[3] The sketch shown in Fig. 2 above is based on this source, on David Spearman[4] and on essays in the *Dictionary of Irish biography.*[5]

Joseph Galbraith, born in Dublin in 1818, entered Trinity College Dublin in 1834, graduated with a Bachelor of Arts in 1839 and became a junior fellow in 1844. He had a highly refined talent for applying mathematics to a

[3] Miguel DeArce, 'The parallel lives of Joseph Allen Galbraith (1818–90) and Samuel Haughton (1821–97): religion, friendship, scholarship and politics in Victorian Ireland', *Proceedings of the Royal Irish Academy* 112C (2011), 333–59.
[4] David Spearman, *Samuel Haughton: Victorian polymath*: a lecture to the National Committee for the History and Philosophy of Science of the Royal Irish Academy on 4th December 2001 (Dublin, 2002).
[5] Desmond McCabe, 'Joseph Allen Galbraith', in James McGuire and James Quinn (eds), *Dictionary of Irish biography* (9 vols, Cambridge, 2009), vol. 4, 524–6. Patrick N. Wyse Jackson, 'Samuel Haughton', in *Dictionary of Irish biography*, vol. 4, 5–6.

wide range of practical problems. He married Hannah Maria Bredon of Co. Cavan in 1850, and they had three sons and four daughters. In 1854 he was appointed Erasmus Smith's professor of natural philosophy. In 1880 he became a senior fellow and was appointed registrar in 1885. In this capacity, he introduced several reforms in the practices of the university.

Samuel Haughton was born in Co. Carlow in 1821 to Quaker parents, but was raised within the established church. He entered Trinity College Dublin aged just sixteen and graduated in 1843. He was elected a fellow in 1844 at the same time as Galbraith. He married in 1848 and was appointed professor of geology in 1851. He was elected a fellow of the Royal Society in 1858. Haughton took up the study of medicine and graduated as a Doctor of Medicine in 1862, aged 40. As registrar of the medical school, he introduced substantial reforms. He is also remembered for devising a method of humane execution by hanging, by lengthening the drop to ensure instant death. In his later career, he was very active in the Royal Zoological Society of Ireland.

Galbraith and Haughton wrote a series of scientific manuals that were remarkably successful. They included basic textbooks on geometry and algebra that were used in Irish schools, together with more advanced books that achieved international circulation. There were twelve titles in all, with total sales of about 300,000 copies and the books remained in print into the twentieth century.[6]

Foucault pendulum experiments in Dublin

Galbraith and Haughton replicated the pendulum experiment in April and May 1851, shortly after Foucault had reported his findings. Their experiment was performed at the engine factory of the Dublin and Kingstown Railway (D&KR) where Samuel's cousin Wilfred Haughton was chief engineer. In 1839 D&KR had acquired the premises of the Dock Distillery at Grand Canal Street. It was converted to a works for the manufacture and maintenance of railway locomotives. The location of the engine factory is indicated on the map in Fig. 3.[7] Grand Canal Street works, as they existed in 1902, are illustrated in Pl. I.[8] The construction bays were accessed by turntables from a siding close to where Grand Canal Dock DART station now stands.

To ensure effective results, a Foucault pendulum should be as long as possible, with a small swing-angle and a heavy bob in order to minimise the effects of air resistance. Engine houses have lofty roofs to allow for the dispersal of smoke and steam. As D&KR was on an embankment, the engine rooms behind the arches in Pl. I were on the first floor of the building. Thus, there was ample headroom and a pendulum of over 35 feet in length could be accommodated.

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[6] DeArce, 'The parallel lives of Galbraith and Haughton', 340.

[7] Rob Goodbody, *Irish Historic Towns Atlas No. 26, Dublin, Part III, 1756 to 1847* (Dublin, 2014), detail from Map 4.

[8] K. A. Murray, *Ireland's first railway* (Dublin, 1981).

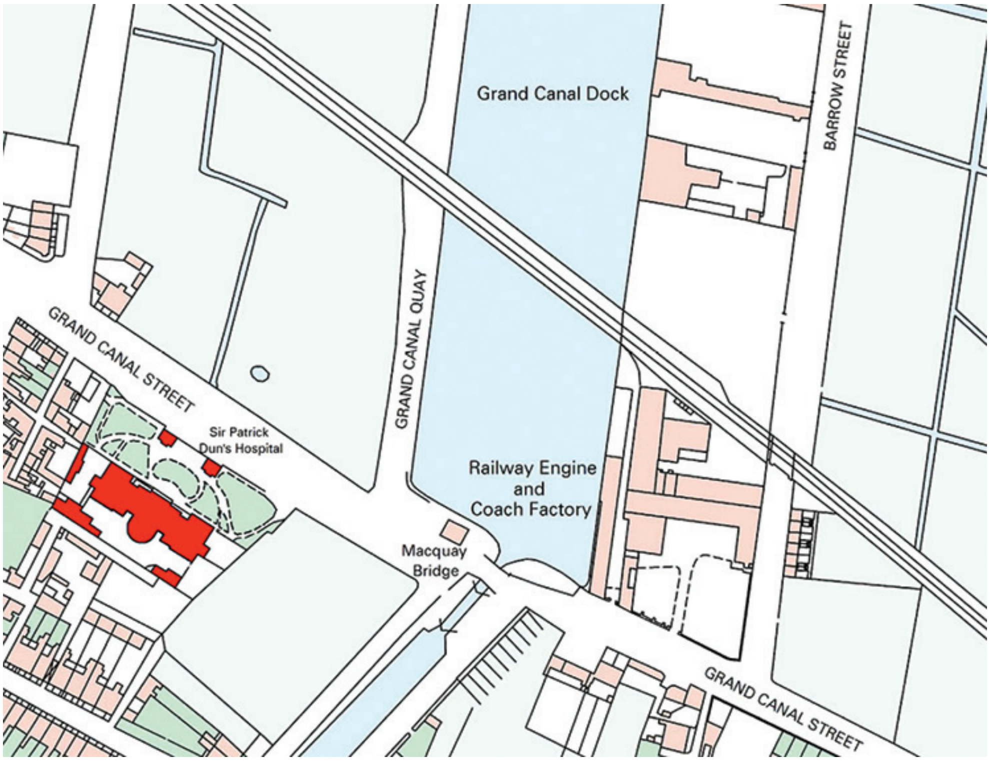


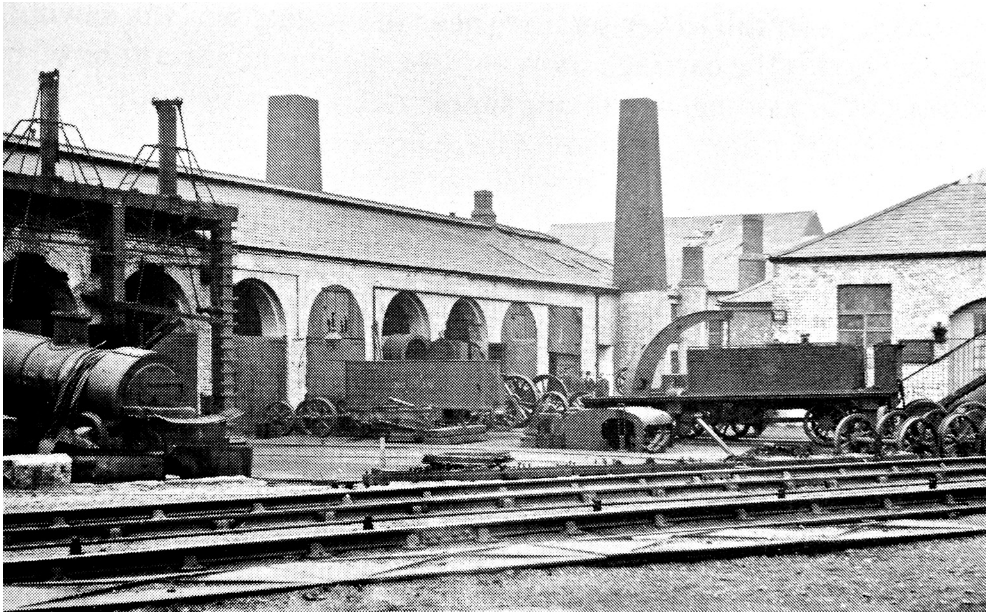
FIG. 3—Detail from a map of Dublin, 1843–7, showing the location of the locomotive factory at Grand Canal Dock. Extract from Map 4 in Rob Goodbody, *Irish Historic Towns Atlas, no. 26, Dublin, Part III, 1756 to 1847* (Dublin, 2014).

Details of the apparatus

At a meeting of the Royal Irish Academy in May 1851, Samuel Haughton presented an account of the experiments that he had made together with his cousin Wilfred and Joseph Galbraith 'to determine the azimuthal motion of the plane of vibration of a freely suspended pendulum'.[9] The pendulum they used had a spherical bob of iron, weighing 30 pounds, with a downward spike attached to indicate the position. It was initially set swinging in a north–south plane. Below the bob stood a table on which a graduated circle allowed the azimuth angle to be read off. 'The direction of vibration could be determined with considerable accuracy.'[10] The minor axis of the elliptical path of the bob could also be estimated with precision.

Various modes of suspending the pendulum were tried. The most satisfactory was a length of multistrand silk fibre threaded through a hole in a thick metal mounting plate and attached to a long wire of copper or steel. Following preliminary testing, six experiments were carried out, generally lasting between 15 and 20 hours, with the azimuth being recorded every

[9] Joseph Galbraith and Samuel Haughton, 'On experiments made in Dublin to determine the azimuthal motion of the plane of vibration of a freely suspended pendulum', *Proceedings of the Royal Irish Academy* 5 (1851), 117–25.
[10] Galbraith and Haughton, 'On experiments made in Dublin', 119.



Pl. I—The engine factory of the Dublin & Kingstown Railway at Grand Canal Street. Illustration from K. A. Murray, *Ireland's first railway* (Dublin, 1981). Photo courtesy of the Irish Railway Record Society.

20 minutes. In the final experiment, with silk fibre suspension, a full rotation of 360° was achieved after an elapsed time of 28 hours 26 minutes.

The main results

Foucault had derived the theoretical result that the rate of angular precession of the plane of oscillation should equal the Earth's angular velocity multiplied by the sine of the latitude. For Dublin, this implies precession through a full circle in about 30 hours. The clockwise precession was observed by Galbraith and Haughton, but the angular velocity of the apsides (azimuth angles of maximum swing) was not constant as suggested by theory. They considered several possible causes for the discrepancy, concluding that 'the variable part of the observed motions is due to instrumental errors'.

Galbraith and Haughton also analysed the effects of ellipticity of the trajectory and derived a mathematical expression for the precession due to this effect.[11] For a pendulum of length ℓ tracing an area A, the precession for each orbit is $3A/4\ell^2$. For typical values of the experiments, and with reasonable precautions when starting the pendulum, this amounts to about half a degree per hour, so it is quite small compared to the change of approximately 10° per hour due to the Earth's rotation for a pendulum in middle latitudes.

The pendulum length for the experiments of Galbraith and Haughton was 35.4ft and its (initial) swing length was 4ft. Thus, the amplitude (swing-angle) was about 6.4°. The bob was an iron sphere, 30lb in weight. The

[11] Joseph A. Galbraith and S. Haughton, 'On the apsidal motion of a freely suspended pendulum', *London, Edinburgh and Dublin Philosophical Magazine* 2 (1851), 134–9.

theoretical precession rate at the latitude of Dublin is 12.07°/h. The mean rate observed in the experiments was 11.9°/h, which is surprisingly accurate considering the many possible sources of error. According to an article in the *Philosophical Magazine*, 'Messrs. Galbraith and Haughton. . .have pursued their research with all imaginable precautions'.[12] Their impressive results confirm this assessment.

The sine factor: kinematic explanation

The observed change in the swing-plane of the pendulum is often stated to be due to the Earth turning beneath it. This is an oversimplification. The turning rate for a pendulum at the North Pole is one revolution per day. At other locations, the turn is proportional to the sine of the latitude (a detailed analysis is given in Appendix B). Thus, after one day, the swing-plane does not return to its original position. At 50°N, the period is 24 h/sin 50° = 31.33h. The mathematical term for this phenomenon is 'anholonomy', and it has been a source of confusion ever since Foucault's demonstration.

A simple way of explaining the turning of the swing-plane as a beat phenomenon was given by G. I. Opat.[13] In fact, a similar explanation was published 140 years earlier by the Irish mathematician Matthew O'Brien.[14] The quasi-linear motion of the bob is considered as a combination of two counter-rotating conical motions with equal amplitudes but slightly different frequencies.

Figure 4 shows the geometric configuration of the pendulum on the spherical Earth. For small amplitude motion, the bob remains close to a plane tangent to the Earth at the equilibrium point and we can approximate the dynamics by motion in this plane. There are two forces acting on the pendulum bob: the gravitational force and the tension in the suspending wire (Fig. 5). Gravity $\mathbf{g}$ acts vertically downwards, but tension $\mathbf{T}$ has both vertical and horizontal components $\mathbf{T}_Z$ and $\mathbf{T}_H$. When we allow for the rotation of the Earth there is an additional force, the Coriolis force. In the Northern hemisphere, this deflects the moving bob to the right. Thus, Newton's law for the horizontal motion is

$$\text{mass} \times \begin{pmatrix} \text{Centripetal} \\ \text{Acceleration} \end{pmatrix} = \begin{pmatrix} \text{Tension} \\ \text{Force} \end{pmatrix} + \begin{pmatrix} \text{Coriolis} \\ \text{Force} \end{pmatrix}$$

In the Northern hemisphere, the (small) Coriolis force acts outwards for anticlockwise motion (Fig. 6a) and inwards for clockwise motion (Fig. 6b). Thus, for anticlockwise motion, the Coriolis force *opposes* the tension while for clockwise motion it *reinforces* it. As a result, the centripetal acceleration is

[12] *London, Edinburgh and Dublin Philosophical Magazine* 1 (1851), 562.
[13] G. I. Opat, 'The precession of a Foucault pendulum viewed as a beat phenomenon of a conical pendulum subject to a Coriolis force', *American Journal of Physics* 59 (1991), 822–3.
[14] Matthew O'Brien, 'On symbolic forms derived from the conception of the translation of a directed magnitude', *Philosophical Transactions of the Royal Society* 142 (1852), 161–206.

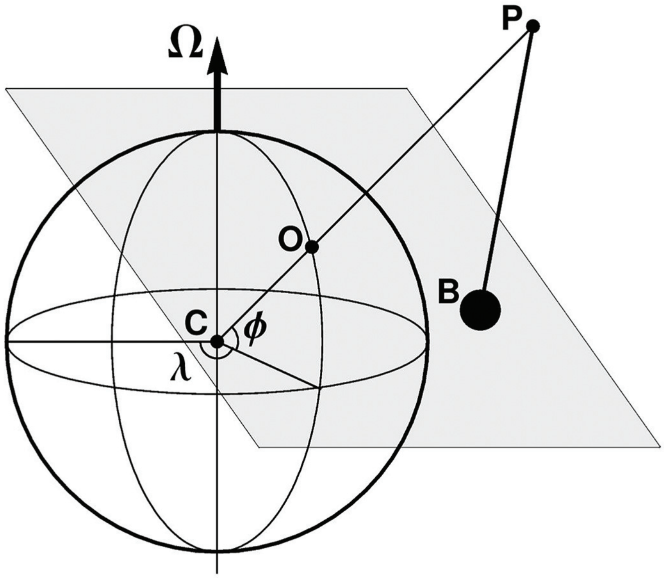


FIG. 4—Geometry of the Foucault pendulum. **C**: Centre of Earth, **P**: point of suspension, **O**: subpoint of pendulum, **B**: pendulum bob, **Ω**: Angular velocity vector of Earth, ϕ: latitude and λ: longitude. The plane tangent to the Earth at the subpoint is shown in grey.

slightly smaller for anticlockwise motion. For conical motion, the centripetal acceleration is proportional to the square of the frequency. Consequently, the frequency is slightly smaller for anticlockwise than for clockwise motion. This leads to a slow clockwise precession of the swing-plane. A more complete dynamical argument is presented in Appendix B.

William Tobin[15] reviewed several other attempts to explain the observed turning angle. Numerous studies have aimed to explain the sine factor in simple terms. However, there is really no mystery; one cannot do better than quote Horace Lamb: 'The simple view of the matter is that the Earth is rotating about the vertical at the rate $\Omega \sin\phi$, in the positive direction, beneath the pendulum'.[16]

Hypocyclic orbit

As we have seen, there is a precessional effect associated with the area traced out by the bob as it completes each oscillation. This effect can confuse the signal due to the Earth's rotation, so it must be minimised by careful choice of the initial conditions. If we set $r_+ = r_-$ (see Appendix B), the bob trajectory resembles a multi-petalled flower (Fig. 7a). If we set $\omega_+ r_+ = -\omega_- r_-$, the initial velocity of the bob vanishes and there is a cusp each time the swing

[15] William Tobin, *The life and science of Léon Foucault* (Cambridge, 2003).
[16] Horace Lamb, *Higher mechanics* (Cambridge, 1929, second edn., reprinted 2009). Here Ω is the magnitude of the Earth's angular velocity and ϕ is the latitude.

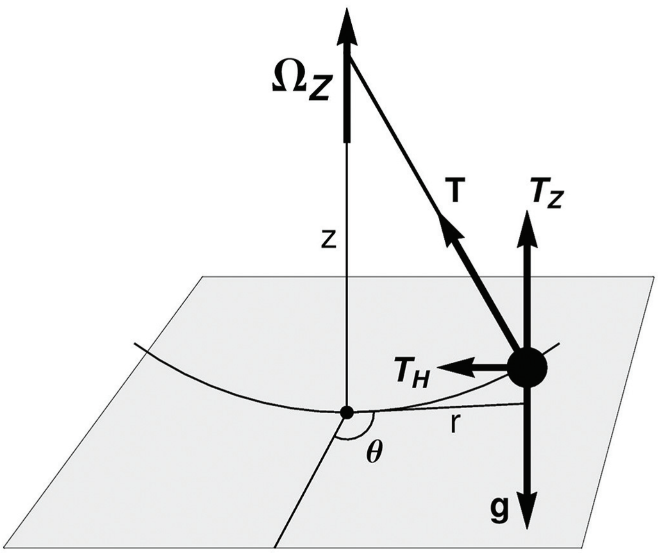


FIG. 5—Foucault pendulum in tangent plane geometry. The forces are the tension **T** with vertical and horizontal components T_Z and T_H and gravity **g**. Local cylindrical coordinates are (r, θ, z). The vertical component of the Earth's angular velocity is $\mathbf{\Omega}_Z$.

reaches a maximum.[17] This solution is shown in Fig. 7b. In the original experiment at the Panthéon, the pendulum was started by burning a thread that held the bob steady.

Conclusion

A much later rerun of the pendulum experiment was performed by the renowned physicist George Francis FitzGerald. In a letter to *Nature*, W. R. Westropp Roberts[18] reported that a Foucault pendulum had been erected in the

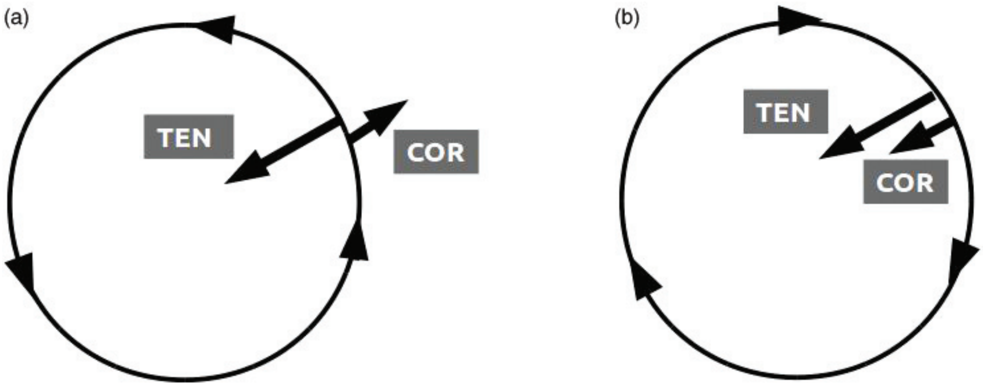


FIG. 6—Horizontal components of forces acting in the case of conical motion. (a) anticlockwise. (b) clockwise motion. TEN: tension force. COR: Coriolis force.

[17] T. J. I'A. Bromwich, 'On the theory of Foucault's pendulum, and of the gyrostatic pendulum', *Proceedings London Mathematical Society* 2:1 (1913), 222–35.
[18] W. R. Westropp Roberts, 'A Foucault pendulum at Dublin', *Nature* 51 (1895), 510–11.

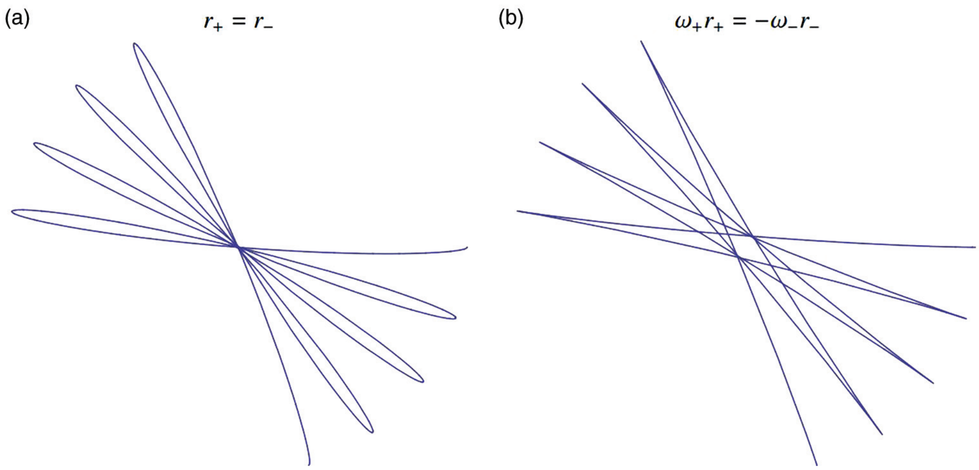


FIG. 7—Solution when $\Omega_Z = \omega_0/20$. (a) $r_+ = r_-$. (b) $|\omega_+|r_+ = |\omega_-|r_-$.

'New Building' at Trinity College Dublin (now the botany building). He wrote that 'Prof FitzGerald and I secured a length of 45 feet' of wire and attached to it a bob weighing 16lb. The pendulum was started by burning a thread 'in the usual manner'. An image of the shadow of the supporting wire was projected through a lens onto a screen 32ft distant, in focus when the wire was closest to the lamp. This served to amplify the signal of precession greatly. Initially, the shadow corresponded with a vertical line drawn on the screen. Within five minutes, it had moved distinctly to the left, indicating a clockwise precession. Westropp Roberts claimed the experiment was a 'complete success' but, in view of the absence of any quantitative information about the rate of precession, this claim appears over exuberant.

To conclude, the author quotes from an article in 1907 in the *British Medical Journal* describing the new science laboratories at Trinity College Dublin:

> The great height available…renders it possible to demonstrate many experiments which cannot be shown in an ordinary lecture room; for instance, it has been found possible, by means of a Foucault pendulum…to enable a large class, seated in this theatre, to appreciate the rotation of the Earth.[19]

Acknowledgements

I wish to thank Estelle Gittins of Trinity College Dublin Manuscripts and Archives Research Library for facilitating access to the diaries of Joseph Galbraith. Thanks also to Siobhán FitzPatrick for guidance on the records in the library of the Royal Irish Academy. I am grateful for discussions with Miguel DeArce and Patrick Wyse Jackson of Trinity College Dublin. Thanks to Rob Goodbody for drawing my attention to the account by K. A. Murray of Ireland's first railway. Finally, thanks to the mathematical reviewer, whose comments assisted me greatly in improving the presentation of this paper.

[19] 'New science libraries, Trinity College Dublin', *British Medical Journal*, 26 October 1907, 1168–70.

Appendix A

Extracts from Joseph Galbraith's diary for 1851

The following are some extracts from the diary of Joseph Galbraith for the months of April to July 1851.[1]

17 April: With SH [Samuel Haughton] and Wilfred H[aughton] at Ringsend arranging Foucault's experiment for tomorrow
18 April: At Ringsend observing pendulum with S^{ℓ} and Wilf H. Dined there. Observation lasted 8 hours
20 April: [EASTER SUNDAY] After chapel to Wilf Haughton's at Ringsend Docks
21 April: Pendulum experiments at Ringsend
22 April: Working at pendulum equations
23 April: ditto
24 April: Set pendulum going at 9.20PM. SH and WH sat observing till 2.20AM [next morning]
25 April: I took up observing at 4.40AM and remained till 2PM
26 April: I took up observations at 7AM and remained till 1 app.
28 April: At railway works all day preparing diagrams and tabulating. SH and I made a statement of experiments at the Royal Irish Academy
30 April: Prepared diagrams for [Humphrey] Lloyd and brought them to him

The entries referring to the pendulum continued into May, beginning with mention of a method of suspending the pendulum using multiple silk fibres near the mounting plate.

1 May: Experiment with silk suspension at Railway. . . .after dinner to Railway Works
2 May: Went to Railway after tea. S^{ℓ} and Wilfred watching a 2^{nd} Silk exper.
14 May: Sent round account of experiments re dependulum(?). $Illus^{s}$ sent to James
17 May: I took(?) Butcher and Ingram down to Ringsend to see the pendulum.
21 May: SH came home with me. He introduced $y\frac{de}{dt}$.

The meaning of the last entry is unclear. It may refer to a damping term (proportional to a first derivative) in the analysis. There are also entries on several other days (5–8, 16, 22, 24, 26, 27 and 28 May) indicating that Galbraith was at the Railway Works. On 30 May he has the curious entry 'Not at Railway'. The following day he writes 'Walked with S^{ℓ} H at Docks'. This appears to signal the end of the experiments.

[1] Manuscripts and Archives Research Library, Trinity College Dublin, MS 3826, Diary of Joseph Galbraith for April and May 1851.

On 23 May Galbraith recorded that 'At 20 minutes of 10 o'clock dear Hannah was safely delivered of a daughter'. Hannah was Galbraith's wife, whom he had married the previous July.

1 June: Sent investigat^{n} of motion of apsis to S$^{\ell}$ H
23 June: Sent communicat^{n} of apsidal mot^{n} to Philosophical Mag. At R. I. Acad^{y} communicated a short notice of apsidal motion
24 June: Left Abstract at [John Hewitt] Jellett's

On 14 July he writes that he is 'working at apsidal motion'. This is interesting, as the analysis by Galbraith and Haughton that appeared in the August 1851 issue of the *Philosophical Magazine*[2] was dated 14 July 1851. In late July Galbraith was in London and in August he was touring around Ireland, so he would have had no opportunity to do any further work on the pendulum.

[2] Joseph A. Galbraith and Samuel Haughton, 'On the apsidal motion of a freely suspended pendulum', *London, Edinburgh and Dublin Philosophical Magazine* 2 (1851), 134–9.

Appendix B

Dynamical analysis of the Foucault pendulum equations

It is convenient to write the equations in a frame of reference rotating with the Earth. The relationship between the absolute and relative motions is given by the equation

$$\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\mathrm{A}} = \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{A}$$

where $\mathbf{A}$ is an arbitrary vector, $\boldsymbol{\Omega}$ is the angular velocity of the Earth and subscripts A and R indicate values in the absolute and rotating frames. The equation was first expressed in this form by Matthew O'Brien.[1] Taking the vector $\mathbf{A}$ to be the position vector $\mathbf{r}$, we get

$$\mathbf{v}_{\mathrm{A}} = \mathbf{v}_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{r}$$

where $\mathbf{v}_{\mathrm{A}}$ is the velocity in the non-rotating frame and $\mathbf{v}_{\mathrm{R}}$ is the velocity in the rotating frame.

It is now straightforward to write down the Lagrangian for the pendulum in rotating coordinates. Assuming unit mass ($m = 1$), the kinetic energy is

$$K = \frac{1}{2}\mathbf{v}_{\mathrm{A}} \cdot \mathbf{v}_{\mathrm{A}} = \frac{1}{2}(\mathbf{v}_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot (\mathbf{v}_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{r}).$$

In local cylindrical coordinates, $\mathbf{v}_{\mathrm{R}} = \left(\dot{r}, r\dot{\theta}, \dot{z}\right)$. For small amplitudes, $|\dot{z}| \ll |\dot{r}|$ so the term $\dot{z}^2$ in K can be dropped. If we assume that $\mathbf{r}$ and $\mathbf{v}_{\mathrm{R}}$ are horizontal, only the vertical component of the Earth's angular velocity, $\boldsymbol{\Omega} \cdot \mathbf{k} = |\Omega| \sin \phi \equiv \Omega_z$, occurs in the term $\mathbf{v} \cdot \boldsymbol{\Omega} \times \mathbf{r}$. Assuming a central force represented by a potential $V(r)$, we have

$$L = \frac{1}{2}\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2 + 2\Omega r^2\dot{\theta} + \Omega^2 r^2\right] - V(r)$$

The centrifugal term $\Omega^2 r^2$ due to the Earth's rotation is negligible compared to that arising from the motion, $\left(r\dot{\theta}\right)^2$, so it can be dropped. Then the Euler-Lagrange equations are

$$\left(\ddot{r} - r\dot{\theta}^2\right) - 2\Omega_Z r\dot{\theta} = -\partial V/\partial r$$
$$\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) + 2\Omega_Z \dot{r} = 0$$

The bob moves on the sphere $(z - \ell)^2 + r^2 = \ell^2$. Linearising this for small z, the potential energy is $V(r) = gz \approx \frac{1}{2}(g/\ell)r^2$. We seek conical motions, where

[1] Peter Lynch, 'Matthew O'Brien: an inventor of vector analysis', *Bulletin of the Irish Mathematical Society* 74 (2014), 81–8.

the bob remains at a constant height and distance from the axis through the point of suspension. Then $\dot{r} = \ddot{\theta} = 0$ and the equations reduce to

$$-r\dot{\theta}^2 - 2\Omega_z r\dot{\theta} = -(g/\ell)r$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[r^2\left(\dot{\theta} + \Omega_z\right)\right] = 0$$

The second equation represents conservation of angular momentum. Writing $\omega = \dot{\theta}$ and $\omega_0^2 = g/\ell$, the first equation reduces to a quadratic for ω:

$$\omega^2 + 2\Omega_z\omega - \omega_0^2 = 0.$$

The values of the frequencies of the conical motions are now easily computed. Since $\Omega_z \ll \omega_0$, the approximate solutions of the quadratic are

$$\omega_+ = \omega_0 - \Omega_Z \quad \text{and} \quad \omega_- = -\omega_0 - \Omega_Z$$

for counter-clockwise and clockwise motion, respectively.
Now we consider a combination of the two conical motions:

$$x = \tfrac{1}{2}\left[r_+\cos\left(\omega_+ t - \psi_+\right) + r_-\cos(\omega_- t - \psi_-)\right]$$
$$y = \tfrac{1}{2}\left[r_+\sin\left(\omega_+ t - \psi_+\right) + r_-\sin(\omega_- t - \psi_-)\right]$$

Nothing is lost by setting $\psi_+ = \psi_- = 0$. For two components of equal amplitude $r_+ = r_- = r_0$, the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = r_0\begin{pmatrix} \cos\Omega_Z t \\ -\sin\Omega_Z t \end{pmatrix}\cos\omega_0 t.$$

Both x and y vary with frequency ω_0, representing the rapid oscillations of the pendulum. However, their amplitudes are modulated as the swing-plane turns clockwise with frequency Ω_Z. This solution was shown for $\Omega_Z/\omega_0 = 1/20$ in Fig. 7a.

Appendix C

The gyroscope and gyrocompass

The pendulum experiment is often described as simple, but there are several practical difficulties. Joseph Galbraith and Samuel Haughton reported irregularities in the precession rate. Non-uniformities in the bob and wire cause anisotropy, which results in oscillations of the swing-plane that can mask the effect of the turning Earth. Small transverse motions introduce anisochrony: since the period depends on the amplitude, the transverse motions are slightly faster than the long swing of the pendulum, causing precession of the elliptical trajectory; again, this can corrupt the signal due to the spinning globe. Finally, friction at the bearing and air resistance on the wire and ball damp and distort the motion.

Another difficulty arises from the sine factor. As we have seen, this can be explained in both kinematic and dynamical terms. However, the anholonomy whereby the pendulum does not regain its initial position after a day remains a complication and something of an enigma. Just one year after the pendulum experiment, Léon Foucault devised another mechanical demonstration of the Earth's rotation. He used a device that kept its orientation fixed relative to the stars and so had a 24-hour period relative to the spinning Earth. This was a rapidly rotating heavy wheel, which other scientists had used but which Foucault developed and named the 'gyroscope'.

The angular momentum vector of the gyroscope points along its axis. If there is no moment acting on the system, this vector remains fixed relative to the stars. The device is constricted so that the centre of gravity is at its geometric centre; then the nett moment of all the forces at the bearings vanishes so there is nothing to change the angular momentum. As the Earth rotates, the gyroscope traces out a conical surface, returning to its initial position after a day. So, the complication of the sine factor is removed.