Initialization of a Barotropic Limited-Area Model Using the Laplace Transform Technique

PETER LYNCH

*Royal Netherlands Meteorological Institute, De Bilt, The Netherlands**

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ABSTRACT

The Laplace transform technique of initialization is used to initialize the data for a barotropic forecasting model over a limited area. The initialization is successful in suppressing high-frequency oscillations during early forecast hours. It has negligible effect upon the resulting 24-hour forecast.

A variation of the linearization, wherein the Coriolis parameter is held constant, is investigated. It is found that the fields which result after a single nonlinear iteration of the modified scheme are almost identical to those resulting from the more general scheme.

1. Introduction

Initialization for limited-area models is a topic of considerable current interest. The nonlinear normal mode method (Machenhauer, 1977; Baer, 1977) provides a very effective method of defining initial conditions for global and hemispheric models. Adaptation of this method for limited-area models poses serious difficulties; it is difficult to determine appropriate normal modes, especially if the horizontal variables do not separate, and to allow for general boundary conditions. An alternative method, which uses a modified inversion formula for the Laplace transform, was proposed by Lynch (1985) and was shown to be effective in controlling high-frequency oscillations in a simple one-dimensional model with periodic boundary conditions. Although the method is equivalent to nonlinear normal mode initialization its application does not require explicit knowledge of the model normal modes.

In the present study the new initialization method is applied in the context of a one-level model over a limited area. The shallow water equations which govern the flow have linear solutions of both low-frequency rotational and high-frequency gravity–inertia wave types. The initial data for the model is taken from the standard 500 mb analysis. Since this analysis normally contains spuriously large gravity-wave components, the resulting forecast exhibits large amplitude high frequency oscillations during the early forecast hours. These oscillations are gradually dissipated by a light diffusive damping which is applied near the boundaries of the forecast area.

To control the initial spurious oscillations the analysis (at 500 mb) is initialized using the Laplace transform method. The forecast from the balanced initial fields evolves very smoothly, without any initial shock or subsequent large oscillations. In the specific case considered, the analysis for 0000 GMT 22 November 1982, the rms (root-mean-square) difference between the original and balanced analysis is about 10 m, with a maximum difference of less than 40 meters. The rms difference in wind speed is about $2\frac{1}{2}$ m s^{-1}. The 24-hour forecasts resulting from the two analyses are virtually identical: the maximum differences in height and wind are 4 m and $\frac{1}{2}$ m s^{-1}. Thus, the initialization process does not affect the final forecast, but it controls the high frequency oscillations by removing spuriously large gravity-wave components from the analysis.

The initialization scheme is varied by holding the Coriolis parameter constant in the linearization and including its variation with the nonlinear terms. The results after one nonlinear iteration are almost identical to those for the original linearization. Since the modified linear equations have separable horizontal variables, a considerably more economical scheme can be formulated using this linearization.

2. Formulation of the Laplace transform technique

a. Outline of the method

A description of the theoretical basis of the Laplace transform technique of initialization can be found in Lynch (1985) and only an outline is given here. We define a modified inverse Laplace transform operator

$$\mathcal{L}^*\{\hat{f}\} = \frac{1}{2\pi i}\oint_{C^*} e^{st}\hat{f}(s)\cdot ds \qquad (1)$$

* Permanent Affiliation: Irish Meteorological Service, Dublin.

where $\mathbf{C}^*$ is a circle of radius r centered at the origin of the s-plane and $\hat{f}$ is the Laplace transform of f. As shown in the Appendix, the operator $\mathcal{L}^*\mathcal{L}$, acting on a time-varying function $f(t)$, filters out all components with frequency greater than r. This filter may be used to formulate an initialization procedure. Consider a system whose state $\mathbf{X}(t)$ is governed by the nonlinear vector equation

$$\frac{d\mathbf{X}}{dt} + \mathbf{LX} + \mathbf{N}(\mathbf{X}) = \mathbf{0} \tag{2}$$

where $\mathbf{L}$ is a constant linear operator and $\mathbf{N}$ is a nonlinear vector function. We wish to adjust the initial conditions $\mathbf{X}^0$ to ensure the slow evolution of the system. As a first approximation the nonlinear terms are ignored and the Laplace transform of (2) is written

$$\hat{\mathbf{X}} = (s\mathbf{I} + \mathbf{L})^{-1}\mathbf{X}^0.$$

Applying $\mathcal{L}^*$ with $t = 0$ we get the first estimate of the balanced initial conditions $\mathbf{X}_1{}^0$. Now let $\mathbf{X}_n{}^0$ be the nth estimate; we approximate the nonlinear term by $\mathbf{N}(\mathbf{X}_n{}^0)$, assumed constant; then, the next estimate of the transformed solution is given by

$$\hat{\mathbf{X}}_{n+1} = (s\mathbf{I} + \mathbf{L})^{-1}[\mathbf{X}_n{}^0 - \mathbf{N}(\mathbf{X}_n{}^0)/s] \tag{3}$$

and the $(n + 1)$th estimate of the balanced initial conditions is

$$\mathbf{X}_{n+1}{}^0 = \mathcal{L}^*\{\mathbf{X}_{n+1}{}^0\}|_{t=0}$$

with the value of r chosen to lie between the low frequencies which we wish to preserve and the high frequencies which we wish to eliminate. Normally only one nonlinear iteration is required in the case of a one-level model.

The contour $\mathbf{C}^*$ is approximated by an inscribed polygon and the integral in (1) is calculated by evaluating the integrand at the center, s_n of each side Δs_n and forming a sum as follows

$$\oint_{\mathbf{C}^*} \hat{f}(s)\cdot ds \approx \sum_{n=1}^{N} \hat{f}(s_n)\cdot \Delta s_n. \tag{4}$$

A constant c has Laplace transform c/s. It is straightforward to show that the approximation (4) with $\hat{f}(s) = c/s$ overestimates c by a factor

$$\kappa = \tan(\pi/N)/(\pi/N)$$

where N is the order of the polygon. This is significant for small N, so we therefore correct the sum in (4) by dividing by κ. This gives excellent results with as few as eight points around $\mathbf{C}^*$ (an octogon).

When the original function $f(t)$ is real we have $\hat{f}(\bar{s}) = \overline{\hat{f}(s)}$ and it is easy to show that

$$\frac{1}{2\pi i}\oint_{\mathbf{C}^*} \hat{f}(s)\cdot ds = \frac{1}{\pi}\int_{C_l} \mathrm{Im}[\hat{f}(s)\cdot ds] \tag{5}$$

where Im[·] is the imaginary part and $\mathbf{C}_l$ is the upper half of $\mathbf{C}^*$, traversed anticlockwise. Since the dependent variables $\mathbf{X}(t)$ are real in the present problem, the use of (5) halves the work required, and only four evaluations of the transformed function on $\mathbf{C}_l$ are needed to give satisfactory results.

b. Description of the forecasting model

The model equations can be written in the form

$$\frac{\partial\Phi}{\partial t} + (1/\epsilon)\nabla\cdot\mathbf{V} = -N_\Phi \tag{6}$$

$$\frac{\partial u}{\partial t} - fv + \frac{\partial\Phi}{\partial x} = -N_u \tag{7}$$

$$\frac{\partial v}{\partial t} + fu + \frac{\partial\Phi}{\partial y} = -N_v. \tag{8}$$

They have been nondimensionalized using length- and time-scales a and $(2\Omega)^{-1}$. The nonlinear terms N_Φ, N_u and N_v have been collected on the right hand side, $\epsilon = (2\Omega a)^2/\bar{\Phi}$ and all other notation is conventional. The mean height of the 500 mb surface is chosen as 5560 m, so that $\epsilon = 15.7$.

The equations are integrated over a limited area with a transformed latitude/longitude coordinate system: the North Pole of the transformed grid is at 30°N, 150°E, obtained by rotating the geographic grid through $\lambda_0 = -30°$ about the geographic polar axis and then through $\phi_0 = 60°$ in the plane of 30°W–150°E. In the transformed (λ, ϕ) coordinates the Coriolis parameter is of the form

$$f = (\cos\phi_0)\sin\phi + (\sin\phi_0)\cos\lambda\cos\phi$$

and is thus a function of both coordinates of the new system. This seriously complicates the linear analysis by making the horizontal variables nonseparable. (Equations relating the regular and transformed latitude/longitude can be found in, for example, Verkley, 1984).

The integration area is spanned by 81 × 51 gridpoints, with geopotential and winds being specified at alternate intersections of a 1° × 1° mesh (Arakawa *E*-grid). Thus, the grid spacing between like points is 157 km at the model equator. The area covered can be seen by reference to Fig. 1.

The timestep is fixed at $\Delta t = 450$ s for both advection and adjustment terms; this ensures stability of the gravity waves. A split explicit method is used to integrate the equations: the advection is handled using a multiply-upstream semi-Lagrangian scheme with biquadratic interpolation (Bates and McDonald, 1982); a forward–backward scheme is used for the gravity wave terms and a trapezoidal (pseudo-) implicit scheme for the Coriolis terms (Mesinger and Arakawa, 1976). The variables on the outermost boundary line are held constant and those on the first inner line are evaluated at each timestep by linear interpolation

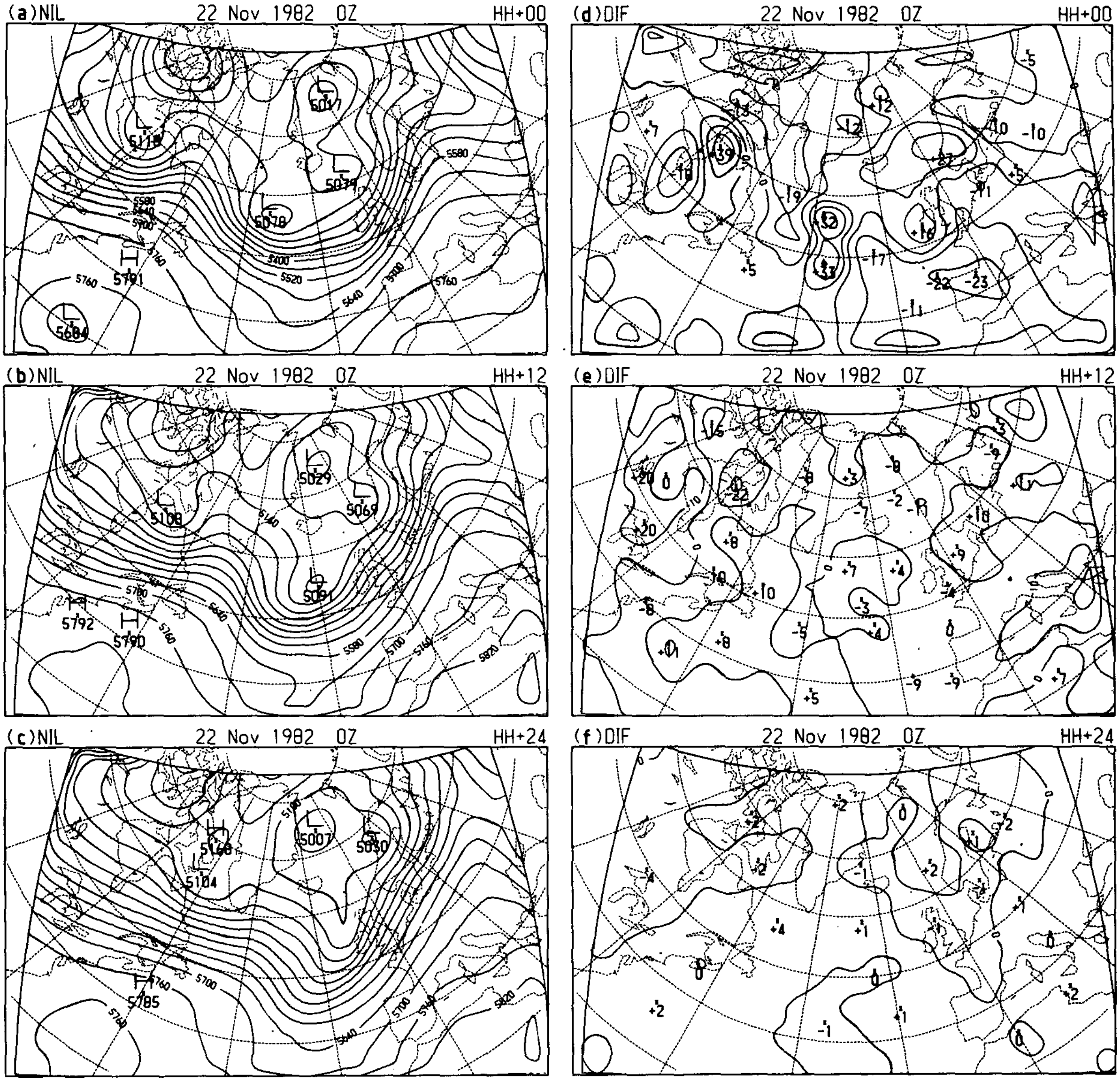

FIG. 1. (a) Original 500 mb analysis valid at 0000 GMT 22 November, 1982; (b), (c) 12 and 24 hour forecasts starting from this analysis; (d) Initialized minus original analysis; (e), (f) differences at 12 and 24 hours between forecasts from original and initialized analyses. (Units are meters).

from the four surrounding points. Bilinear interpolation is also used for the Lagrangian advection scheme on the next three lines, which results in some damping. In addition, light diffusive damping is applied over the five outermost lines of the grid.

c. Application of the initialization method

The Laplace transform of (6)–(8) may be written

$$s\hat{\Phi} + (1/\epsilon)\nabla\cdot\hat{\mathbf{V}} = \Phi^0 - \hat{N}_\Phi \quad (9)$$

$$s\hat{u} - f\hat{v} + \hat{\Phi}_x = u^0 - \hat{N}_u \quad (10)$$

$$s\hat{v} + f\hat{u} + \hat{\Phi}_y = v^0 - \hat{N}_v \quad (11)$$

where $\hat{\Phi}$ denotes the Laplace transform of Φ, etc. We discretise the domain and replace spatial derivatives by centered differences in the usual way. The values of Φ, u and v on the "kth row" of the grid are collected in the vector

$$\mathbf{X}_k = (\Phi_{1k}, u_{1k}, v_{1k}, \ldots, \Phi_{Mk}, u_{Mk}, v_{Mk})$$

and the transformed vector $\hat{\mathbf{X}}_k$ is defined in a similar manner. Because of the grid staggering, some boundary points are included in $\mathbf{X}_k$. We assume that these are constant and that points adjacent to the boundaries are defined by interpolation from the surrounding points. The system (9)–(11) may now be written in the form of a set of matrix equations

$$\mathbf{A}_k\hat{\mathbf{X}}_{k-1} + \mathbf{B}_k\hat{\mathbf{X}}_k + \mathbf{C}_k\hat{\mathbf{X}}_{k+1} = \mathbf{D}_k \qquad (12)$$

where $\mathbf{A}_k$, $\mathbf{B}_k$ and $\mathbf{C}_k$ are block–tridiagonal matrices whose elements depend upon the coefficients of the equations and $\mathbf{D}_k$ is a column vector of initial values plus transformed nonlinear terms. The lateral boundary values u_{1k}, u_{Mk}, etc. also occur in the vector $\mathbf{D}_k$. The system (12) is solved using the method proposed by Lindzen and Kuo (1969). The first (linear) step in the initialization is performed after setting the nonlinear terms in $\mathbf{D}_k$ to zero. The system (12) is solved for $\hat{\mathbf{X}}_k(s)$ and this is inverted on the modified contour $\mathbf{C}^*$ to give $\mathbf{X}_1{}^0$, the linearly initialized fields. The nonlinear terms are evaluated either directly from these fields or by making a single timestep forecast. They are incorporated in the forcing vector $\mathbf{D}_k$ and the initialization cycle is repeated as often as required. In the case of a barotropic model a single nonlinear iteration is normally sufficient for convergence.

3. Results

Several test runs have been made with varying grid resolutions and other parameter values. The results described below are for the 500 mb analysis valid at the initial time 0000 GMT 22 November 1982. The grid resolution is 1° × 1° (*E*-grid). The cutoff frequency for the inversion integral (1) is chosen by setting $r = 0.5$; this corresponds to eliminating all components with period less than 24 hours. One linear and one nonlinear iteration of the initialization procedure are applied; it is found that the changes due to a second nonlinear iteration are very small (presumably further iterations would be needed in the baroclinic case where the equivalent depths are progressively smaller). The nonlinear terms may be evaluated directly within the initialization or by making a single timestep forecast of the model. In the present case it was found to be simpler to calculate them directly.

TABLE 1. Root-mean-square (and maximum) changes to the geopotential height and wind fields due to each iteration of the initialization, and to the linear and first nonlinear iterations combined.

		z (m)	u (m s^{-1})	v (m s^{-1})
LIN	rms	16.9	2.56	2.68
	(max)	(77.5)	(15.33)	(12.72)
NL1	rms	10.9	0.23	0.11
	(max)	(46.9)	(0.85)	(0.32)
NL2	rms	0.5	0.02	0.02
	(max)	(1.9)	(0.12)	(0.09)
LIN + NL1	rms	9.5	2.53	2.64
	(max)	(38.9)	(15.32)	(12.78)

TABLE 2. Root-mean-square (and maximum) differences in the geopotential height and wind fields between the original and initialized fields (NL1 minus NIL) and between the 12 and 24 hour forecasts resulting from these fields.

Forecast	Difference	z (m)	u (m s^{-1})	v (m s^{-1})
HH + 00	rms	9.5	2.53	2.64
	(max)	(+38.9)	(−15.32)	(+12.78)
HH + 12	rms	6.1	0.25	0.21
	(max)	(−22.3)	(−0.97)	(−0.78)
HH + 24	rms	1.0	0.11	0.09
	(max)	(+4.1)	(−0.55)	(+0.37)

In Table 1 we show the rms changes (and maximum changes) to the geopotential height and wind fields due to each iteration of the initialization and to the linear and first nonlinear iterations combined. The changes of the height field are quite large for the linear (LIN) and first nonlinear (NL1) iterations; the overall change due to the two (LIN + NL1) is somewhat less. The winds change markedly during the linear step but very little thereafter. In all cases there is hardly any change due to the second nonlinear iteration (NL2); therefore, the results presented below refer to the case of a single nonlinear iteration (LIN + NL1). (NIL indicates results without any initialization).

Figure 1a shows the initial 500 mb height analysis and Figs. 1b and c are the HH + 12 and HH + 24 hour forecasts resulting from this analysis. In Fig. 1d we show the changes to the height field due to the initialization (LIN + NL1), and Figs. 1e and f show the differences, at 12 and 24 hours, between the forecasts starting from the two analyses (initialized minus original). The changes to the analysis, and also after 12 hours, are quite significant. The similarity between the two 24-hour forecasts is remarkable: the maximum height difference is only 4 m, and for practical purposes the forecasts are identical. Further results are presented in Table 2, and they confirm the convergence between the two forecasts.

The effect of initialization on the evolution of the flow is indicated by several diagnostics. In Fig. 2 we show the geopotential at a central point (indicated by a cross in Fig. 6b) resulting from the initial fields and after linear and nonlinear initialization. The reduction of the initial oscillations in the linear case (LIN) is dramatic, and the evolution after NL1 is very smooth. Similar graphs of the evolution of the divergence (at the same central point) tell much the same story: the divergence fluctuates wildly if the initial fields are out of balance (Fig. 3); this fluctuation is controlled by initialization.

The rms divergence and global mean divergent kinetic energy give good overall measures of the noise

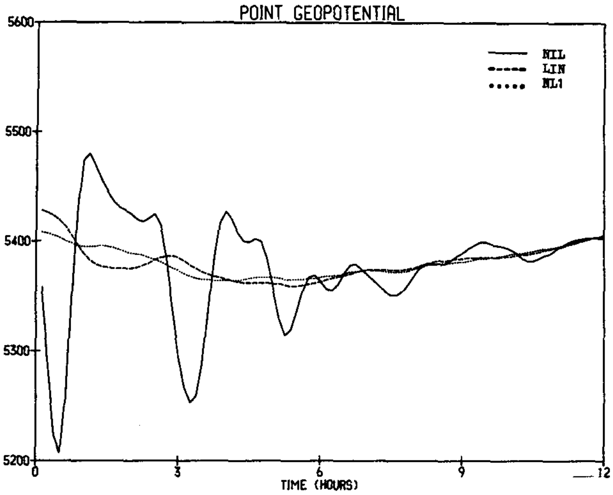

FIG. 2. Geopotential height (m) at the gridpoint $I = 37$, $J = 19$ for the first 12 hours of the forecast starting from uninitialized fields (solid), linearly initialized fields (dashed) and nonlinearly initialized fields (dotted).

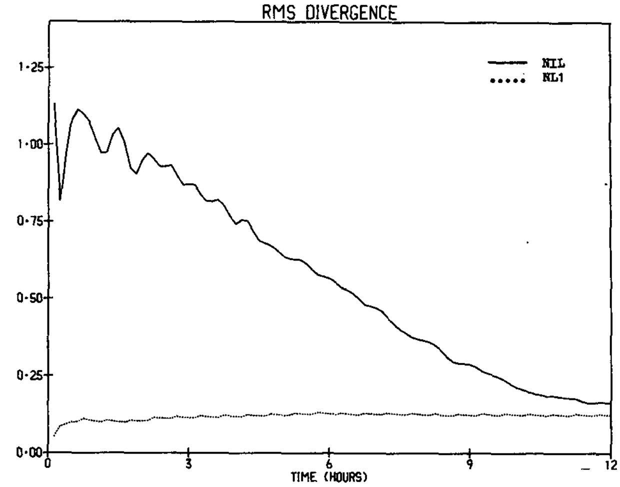

FIG. 4. Root-mean-square divergence (units $\times 10^5$ s^{-1}) for the first 12 hours of the forecast starting from uninitialized fields (solid) and nonlinearly initialized fields (dotted).

in the evolution of the flow. The effects of the initialization upon these quantities are shown in Figs. 4 and 5. In both cases there is a dramatic reduction of the noise in the forecast when the fields are initially balanced.

All the above diagnostics confirm that the initialization (LIN + NL1) is successful in removing spurious oscillations from the early forecast and results in a noise-free evolution of the flow. The remarkable agreement between the 24-hour forecasts *before* and *after* (Fig. 1f) demonstrates that the process is doing precisely what is required: removing high-frequency gravity waves without perturbing the development of the meteorological flow.

4. A simplified linearization

The development of the present method of initialization was guided by an intuitive feeling that it is important to include the full variation of the Coriolis parameter (the β-effect) in the linearized equations. Ballish (1979) showed, in the context of a one-dimensional model, that the omission of the β-terms from the eigenvector analysis leads to larger oscilla-

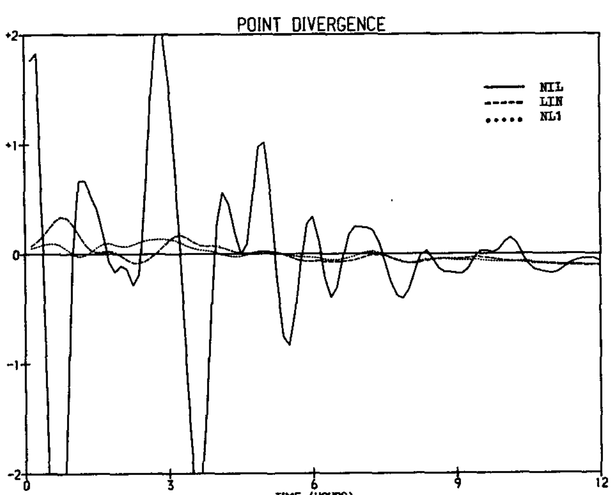

FIG. 3. Divergence at a central point, $I = 37$, $J = 19$ (units $\times 10^5$ s^{-1}) for the first 12 hours of the forecast starting from uninitialized fields (solid), linearly initialized fields (dashed) and nonlinearly initialized fields (dotted).

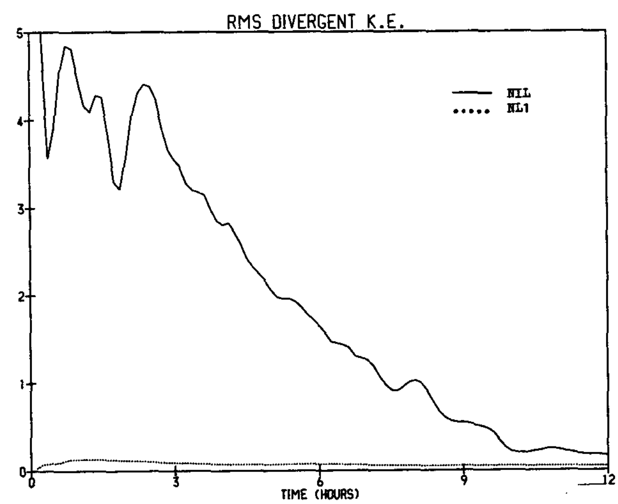

FIG. 5. Root-mean-square divergent kinetic energy per unit mass (m^2 s^{-2}) for the first 12 hours of the forecast starting from uninitialized fields (solid) and nonlinearly initialized fields (dotted).

tions than if they are included. It seemed likely that this would also be true for a more general model. The question is examined below.

The initialization procedure was modified in the following way: The Coriolis parameter f occurring in the linear terms of (7) and (8) was replaced by its mean value f_0, and its variation was accounted for by including the factors $-(f-f_0)v$ and $+(f-f_0)u$ in the nonlinear terms N_u and N_v of these equations. Thus, the nonlinear equations to be solved are unchanged, but they are split into linear and nonlinear parts in a different way.

The difference between the original ($f = f(\lambda, \phi)$) and simplified ($f = f_0$, constant) initialization schemes can be seen from Fig. 6. In Fig. 6a we show the difference in the 500 mb analyses resulting from the original and simplified schemes after linear initialization. Since the linear equations used in the two cases differ, it is hardly surprising that the two analyses differ by as much as 30 m, with an rms difference of 10 m. In contrast to this, Fig. 6b shows that after a single nonlinear iteration the two schemes produce very similar analyses: the maximum difference is 1.6 m, and the rms difference only 0.5 m. The maximum difference in the corresponding wind analyses is only 0.14 m s^{-1}. For practical purposes the two analyses are identical. The noise profiles produced by the forecasts from the two analyses are indistinguishable.

The simplified scheme gives results equivalent to the original method, and results in linearized equations in which separation of the horizontal variables obtains. It is therefore possible with this scheme to develop a more efficient initialization procedure. Furthermore, the amount of disk storage should be considerably reduced. It seems highly probable that the equivalence of the two methods will also hold in the case of a baroclinic model. However, the geographical extent of the analysis area must be taken into account. For example, it is not clear how important the inclusion of the β-terms in the linear equations may become if the area extends to or straddles the equator.

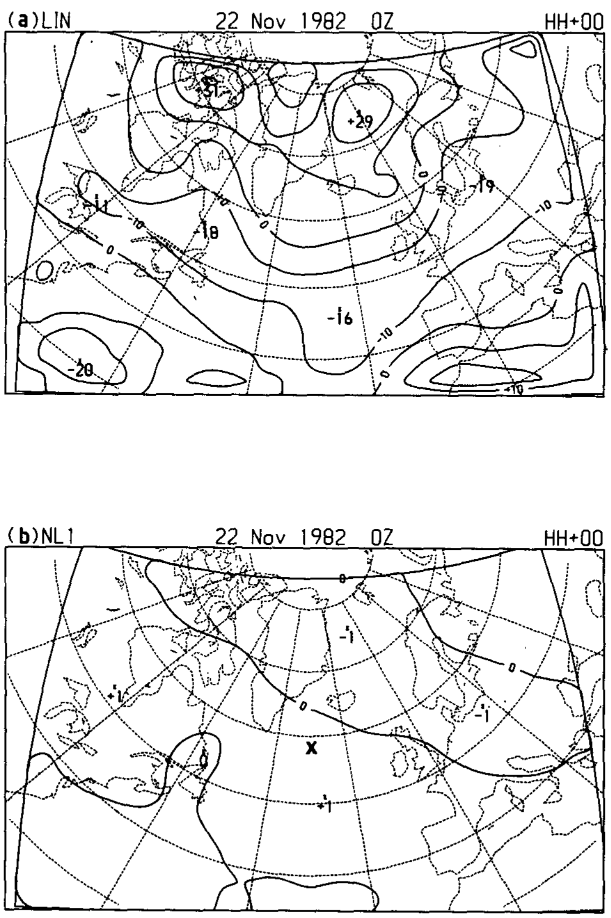

FIG. 6. Difference between the initialized 500 mb analyses resulting from the original and simplified (f constant) schemes, (a) after linear initialization and (b) after one nonlinear iteration. (Units are meters).

5. Summary

The Laplace transform technique has been applied in the context of a one-level limited-area model. The results have shown that the method is capable of removing spurious gravity-wave noise without having any adverse affect on the resulting forecast. There do not appear to be any problems associated with the boundaries. The simplification of the linearization, in which the Coriolis parameter is taken as constant, is found to produce results which are almost identical to those obtained with the more general scheme.

APPENDIX

Laplace Transform Theory

The basic definitions and properties of Laplace transforms needed in this study are summarized, and the method of filtering is described. A good comprehensive guide to the theory, from a practical viewpoint, is Doetsch (1971).

The Laplace transform of a function $f(t)$ of time t is defined as:

$$\mathcal{L}\{f(t)\} = \hat{f}(s) = \int_0^\infty f(t)e^{-st}dt \qquad \text{(A1)}$$

and is a function of the associated complex variable s. Thus, a constant a transforms to a/s. The complex exponential function representing a wavemotion is transformed to an algebraic function

$$\mathcal{L}\{e^{i\omega t}\} = 1/(s - i\omega). \qquad \text{(A2)}$$

Since $\mathcal{L}$ is linear, the superposition of a number of waves may be transformed component-wise:

$$\mathcal{L}\{\sum_{j=1}^{J} a_j e^{i\omega_j t}\} = \sum_{j=1}^{J} a_j/(s - i\omega_j). \qquad \text{(A3)}$$

The higher the frequency ω_j, the further the corresponding pole $s = i\omega_j$ lies from the origin.

If the transform of $f(t)$ is $\hat{f}(s)$ then the time-derivative, $f'(t)$, transforms as

$$\mathcal{L}\{f'(t)\} = s \cdot \hat{f}(s) - f(0) \qquad \text{(A4)}$$

where $f(0)$ is the initial value of $f(t)$. Thus, differentiating in t-space corresponds to algebraic operations in s-space. This is the power of the Laplace transform method: it lowers the level of transcendence of functions and operators. Ordinary differential equations transform to algebraic ones.

A modification of the Fourier theorem gives us the complex inversion formula for the Laplace transform:

$$f(t) = \mathcal{L}^{-1}\{\hat{f}(s)\} = \frac{1}{2\pi i}\int_{\mathrm{C}} \hat{f}(s)e^{st}ds \qquad \text{(A5)}$$

where the contour **C** is parallel to the imaginary axis, and to the right of all singularities of $\hat{f}(s)$. We assume that $\hat{f}(s)$ is meromorphic, that is, analytic except for isolated poles; and that the contour **C** can be completed by an asymptotically large semicircular arc in the left half plane.

The contributions to $f(t)$ in (A5) come from the poles of $\hat{f}(s)$. Since the high-frequency components correspond to poles far from the origin, they can be eliminated by shrinking the contour to a circle **C*** of radius r centered at the origin. This is the motivation for the definition of the operator $\mathcal{L}^*$ [Eq. (1), Section 2a], and we see that $\mathcal{L}^*\mathcal{L}$ acting on $f(t)$ will select the components with frequency $\omega < r$ and filter out the high-frequency components of $f(t)$.

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