Hamiltonian Methods for Geophysical Fluid Dynamics: An Introduction[*]

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Abstract

The value of general Hamiltonian methods in geophysical fluid dynamics has become clear over recent years. This paper provides an introduction to some of the key ideas necessary for fruitful application of these methods to problems in atmosphere and ocean dynamics. Hamiltonian dynamics is reviewed in the context of simple particle dynamics. The non-canonical formalism which is required for fluid dynamics is introduced first in the finite-dimensional case. The Lagrangian and Eulerian formulations of the fluid dynamical equations are then considered, and the method of reduction from Lagrangian to Eulerian form is described. Rotational effects are introduced in the context of the shallow water equations, and these equations are expressed in Hamiltonian form in both Lagrangian and Eulerian variables. Finally, simple balanced systems are derived, in which constraints are imposed on the fluid motion by applying least action principles to Lagrangians modified either by additional terms with Lagrange multipliers or by direct approximation.

Contents

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1 Introduction

The application of general Hamiltonian methods to fluid dynamics has been an area of increasing interest over the past few decades. A number of reviews have appeared, and this paper has borrowed heavily from several of them. We refer in particular to review papers by Salmon (1988a), Shepherd (1990) and Morrison (1998), and to the recent textbook of Salmon (1998). The underlying theory has considerable æsthetic appeal and the methods of generalized analytical dynamics are powerful. A specific example is Noether's Theorem, which provides a connection between continuous symmetries of the Hamiltonian function and conservation laws of the system.

We will examine the application of Hamiltonian theory in its generalized formulation to problems in geophysical fluid dynamics. The canonical form which is applicable to many finite-dimensional systems is not the appropriate framework; indeed, it was the undue emphasis on the canonical equations which hampered progress in Hamiltonian fluid dynamics for so long. A more general formulation is required; this is presented in §2 below. It consists of identifying appropriate generalized coordinates z, a Hamiltonian function $H(z)$ and a symplectic operator J having special algebraic properties. Some of the new ideas are illustrated through the example of the swinging spring, a simple mechanical system with interesting properties (Lynch, 2002a).

There are several advantages associated with the general Hamiltonian formulation. First, Hamiltonian methods are not tied to a particular coordinate system. The freedom to choose coordinates can result in significant simplifications of the equations. Second, due to the relationship between symmetries and conservation laws, approximations which conserve symmetries of the Hamiltonian also retain analogues of the conservation laws of the exact system. Thus, various balance systems may be derived, which have energetics consistent with the systems from which they are derived. Thirdly, the general Hamiltonian framework can be a powerful starting point for perturbation analysis. Approximations may be introduced directly into the system Hamiltonian or Lagrangian function *before* the least action principle is used to obtain approximate equations. The average Lagrangian technique is frequently easier to apply than alternative perturbation techniques.

2 Finite Dimensional Mechanical Systems

2.1 The Canonical Equations

The state of a mechanical system with N degrees of freedom, comprising a finite collection of discrete particles, can be specified by the generalized coordinates $\{q_n, n = 1, 2, \ldots, N\}$ as functions of the time t. The dynamics of the system are determined by the Lagrangian $L = T - V$, the difference between the kinetic and potential energies. The Lagrangian $L = L(q_n, \dot{q}_n, t)$ is a function of the coordinates q_n, the velocities $\dot{q}_n$ and possibly the time t. The evolution of the system may be determined from Hamilton's principle

$$\delta \int_0^{t_0} L\, dt = 0 \tag{1}$$

where the variation δ is the change in the action $\int L dt$ resulting from variations δq_n which vanish at the initial and final times $t = 0$ and $t = t_0$. The solution of this variational problem yields the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0\,. \tag{2}$$

Note that the dynamics is unaffected by scaling of L or addition of an arbitrary constant to it: $L \to \alpha L + \beta$ with α and β constant does not result in any change to (2).

EXAMPLE 1: THE SWINGING SPRING I. The Lagrangian of a spherical elastic pendulum or *swinging spring* may be written

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{Z}^2) - \tfrac{1}{2}k(r - \ell_0)^2 - mgz$$

where m is the mass of the bob, k the stiffness of the spring and g the acceleration due to gravity. The first right-hand term is the kinetic energy, the second is the elastic potential energy and the third is the gravitational potential energy. The coordinates are $(q_1, q_2, q_3) = (x, y, Z)$, cartesian coordinates centered at the point of suspension of the spring, and $r = \sqrt{x^2 + y^2 + Z^2}$. The equations of motion may be written immediately using (2):

$$\ddot{x} + \frac{k}{m}\left(\frac{r - \ell_0}{r}\right)x = 0\,, \quad \ddot{y} + \frac{k}{m}\left(\frac{r - \ell_0}{r}\right)y = 0\,, \quad \ddot{z} + \frac{k}{m}\left(\frac{r - \ell_0}{r}\right)z + g = 0\,.$$

The momentum $\mathbf{p} = (m\dot{x}, m\dot{y}, m\dot{z})$ is obtained from derivatives of the Lagrangian with respect to the velocities.

The generalized momentum is defined as

$$p_n = \frac{\partial L(q,\dot q)}{\partial \dot q_n}\,. \tag{3}$$

In typical conditions (for a non-singular Lagrangian) these equations may be inverted to obtain $\dot q_n$ as functions of p_n and q_n. We assume that this is the case. The Hamiltonian is defined by means of the Legendre transformation

$$H(q,p) = \sum p_n \dot q_n - L(q,\dot q)\,. \tag{4}$$

We assume that (3) has been solved for $\dot q$ so that H is a function of the canonical variables p and q. Hamilton's principle requires that the action

$$\delta \int_0^{t_0} \left\{ \sum p_n \dot q_n - H(q,p) \right\} dt$$

is stationary for independent variations δq_n and δp_n with δq_n vanishing at the end-times. This results in Hamilton's *canonical equations*:

$$\frac{dq_n}{dt} = \frac{\partial H}{\partial p_n}\,, \qquad \frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n}\,. \tag{5}$$

EXAMPLE 2: THE SWINGING SPRING II. For the swinging spring, the Lagrangian of Example 1, approximated to cubic order in the amplitudes, is

$$L = \tfrac{1}{2}\left(\dot x^2 + \dot y^2 + \dot z^2\right) - \tfrac{1}{2}\left(\omega_R^2(x^2+y^2) + \omega_Z^2 z^2\right) + \tfrac{1}{2}\lambda(x^2+y^2)z\,, \tag{6}$$

where x, y and z are Cartesian coordinates centered at the point of equilibrium, $\omega_R = \sqrt{g/\ell}$, $\omega_Z = \sqrt{k/m}$ and $\lambda = \ell_0\omega_Z^2/\ell^2$. For simplicity we assume $m = 1$. The generalized momenta are $(p_x, p_y, p_z) = (\dot x, \dot y, \dot z)$, and the Legendre transformation (4) yields the Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \tfrac{1}{2}\left(\omega_R^2(x^2+y^2) + \omega_Z^2 z^2\right) - \tfrac{1}{2}\lambda(x^2+y^2)z\,,$$

Then the canonical Hamiltonian equations are

$$\begin{aligned}
\dot x &= p_x\,, \qquad & \dot p_x &= -\omega_{\mathrm{R}}^2 x + \lambda x z\,,\\
\dot y &= p_y\,, \qquad & \dot p_y &= -\omega_{\mathrm{R}}^2 y + \lambda y z\,,\\
\dot z &= p_z\,, \qquad & \dot p_z &= -\omega_{\mathrm{Z}}^2 z + \tfrac{1}{2}\lambda(x^2+y^2)\,.
\end{aligned} \tag{7}$$

The case $\omega_{\mathrm{Z}} = 2\omega_{\mathrm{R}}$ is of special interest. In this *resonant* case, there is strong interchange of energy between the swinging and springing motions (Lynch, 2002a,b). We consider the resonant case below.

2.2 The Symplectic Form

If we define a vector $\mathbf{z} = z^i = \{q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N\}$, the full system (5) can be written in vectorial form

$$\dot{\mathbf{z}} = \mathbf{J}\nabla_{\mathbf{z}} H \,.$$

or in tensorial form as

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j}\,, \tag{8}$$

where the matrix $\mathbf{J} = J^{ij}$, called the *symplectic operator*,[1] is defined by

$$\mathbf{J} = J^{ij} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} . \tag{9}$$

The summation convention applies in (8) and below, where repeated indices are summed over their ranges, unless otherwise indicated.

It is straightforward to show that J^{ij} transforms as a second order contravariant tensor which is skew-symmetric

$$J^{ji} = -J^{ij}$$

and satisfies the Jacobi identity

$$J^{im}\frac{\partial J^{jk}}{\partial z^m} + J^{jm}\frac{\partial J^{ki}}{\partial z^m} + J^{km}\frac{\partial J^{ij}}{\partial z^m} = 0 \,.$$

If J is non-singular then, according to Darboux's Theorem (see, e.g., José and Saletan, 1998), it is possible to transform to coordinates for which J assumes the canonical form (9). If J is singular the system is called *non-canonical*. In general, J is a function of the coordinates z. A system is Hamiltonian if we can specify a function H and an operator J, singular or otherwise, which is skew-symmetric and satisfies the Jacobi identity, and such that the evolution is governed by (8). The advantage of the formulation (8) is that it enables us to consider non-canonical dynamical systems. It is in this general form that the methods may be applied to continuous systems such as fluids.

2.3 Poisson Brackets

The Poisson bracket of two functions of state is a bilinear operator defined as

$$\{F, G\} = \frac{\partial F}{\partial q_n}\frac{\partial G}{\partial p_n} - \frac{\partial F}{\partial p_n}\frac{\partial G}{\partial q_n}$$

In terms of z^i and J^{ij} it may be written

$$\{F, G\} = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j} \tag{10}$$

[1] The term *symplectic*, from the Greek for 'intertwined', was introduced in 1939 by Hermann Weyl in his book *The Classical Groups* (Goldstein, *et al.*, 2002).

and this form provides the definition valid for general J. Following from the properties of the symplectic operator J^{ij}, the Poisson brackets are also skew-symmetric

$$\{F,G\} = -\{G,F\}$$

and satisfy the Jacobi identity

$$\{E,\{F,G\}\} + \{F,\{G,E\}\} + \{G,\{E,F\}\} = 0$$

for all state functions E, F and G. The canonical equations may now be written in the form

$$\dot{q}_n = \{q_n, H\}\,, \qquad \dot{p}_n = \{p_n, H\}\,,$$

or, in terms of the variable $z^i = \{q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N\}$,

$$\dot{z}^i = \{z^i, H\}\,.$$

The time evolution of a general state function $F(z) = F(q,p)$ is given by

$$\frac{dF}{dt} = \{F, H\}\,. \tag{11}$$

This is the evolution equation for a general Hamiltonian system.

A general Hamiltonian system consists of a phase-space and two geometric objects, a scalar H and a Poisson bracket $\{\ ,\ \}$ which is a skew-symmetric bilinear operator satisfying the Jacobi identity.

EXAMPLE 3: THE SWINGING SPRING III. We apply the average Lagrangian technique to the swinging spring (see Holm and Lynch, 2002, for details). The solution is assumed to be of the form

$$x = \Re[a(t)\exp(i\omega_R t)]\,, \quad y = \Re[b(t)\exp(i\omega_R t)]\,, \quad z = \Re[c(t)\exp(2i\omega_R t)]\,.$$

The coefficients $a(t)$, $b(t)$ and $c(t)$ are assumed to vary on a time scale which is much longer than the time-scale of the oscillations. If the Lagrangian (6) is now averaged over the fast time, the Euler-Lagrange equations for the modulation amplitudes are

$$i\dot{a} = \kappa a^* c\,, \qquad i\dot{b} = \kappa b^* c\,, \qquad i\dot{c} = \tfrac{1}{4}\kappa(a^2 + b^2)$$

where $\kappa = \lambda/(4\omega_R)$. If we now introduce new dependent variables A_1, A_2 and A_3 defined by

$$A_1 = \tfrac{1}{2}\kappa(a+ib)\,, \quad A_2 = \tfrac{1}{2}\kappa(a-ib)\,, \quad A_3 = i\kappa c\,,$$

the modulation equations take the following form

$$\begin{aligned} \dot{A}_1 &= -A_2^* A_3\,, \\ \dot{A}_2 &= -A_3 A_1^*\,, \\ \dot{A}_3 &= +A_1 A_2\,. \end{aligned} \tag{12}$$

These are the *three-wave equations*.

The three-wave equations conserve the following three quantities,

$$\begin{aligned} H &= (A_1 A_2 A_3^* - A_1^* A_2^* A_3)\,, \\ N &= |A_1|^2 + |A_2|^2 + 2|A_3|^2\,, \\ M &= |A_1|^2 - |A_2|^2\,. \end{aligned}$$

The first, $H = 2i\Im\{A_1 A_2 A_3^*\}$, is the Hamiltonian of the system (see Holm & Lynch, 2002). The second, N, is a measure of the energy of the oscillations and the third, M, is the angular momentum.

Defining $z = (A_1, A_2, A_3, A_1^*, A_2^*, A_3^*)$, the system (12) together with its complex conjugate can be written in canonical form (8) with J given by (9). Alternatively, if we write $A_n = A_{n\mathrm{R}} + iA_{n\mathrm{I}}$ and define $p = (A_{1\mathrm{R}}, A_{2\mathrm{R}}, A_{3\mathrm{R}})$ and $q = (A_{1\mathrm{I}}, A_{2\mathrm{I}}, A_{3\mathrm{I}})$, then (12) are in the canonical form (5).

2.4 Noether's Theorem

There is an intimate relationship between symmetry and invariance of dynamical systems. This was first elucidated by Emmy Noether (1882–1935). Symmetry is a geometric property in which quantities remain unchanged under coordinate transformation. Invariance is an algebraic or analytical property whereby 'integrals of the motion' are constant along system trajectories. We consider a continuous family of coordinate transformations $q_n(t) \to q_n(\alpha, t)$ parameterized by a single quantity α, such that $q_n(t) = q_n(0, t)$. For example, consider a translation along a single direction, $q_k(\alpha) = q_k + \alpha$ (for notational simplicity, we will drop the explicit dependence on t). We assume that the Lagrangian is not altered by the transformation, $L[q_n(\alpha), \dot{q}_n(\alpha)] = L[q_n, \dot{q}_n]$, which implies a symmetry of the system. Thus,

$$\frac{d}{d\alpha} L[q_n(\alpha), \dot{q}_n(\alpha)] = \frac{\partial L}{\partial q_n(\alpha)} \frac{\partial q_n(\alpha)}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_n(\alpha)} \frac{\partial \dot{q}_n(\alpha)}{\partial \alpha} = 0\,.$$

Since $q_n(\alpha)$ is a solution of Lagrange's equations for any α, we also have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n(\alpha)} - \frac{\partial L}{\partial q_n(\alpha)} = 0\,.$$

Combining these two equations gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n(\alpha)}\right) \frac{\partial q_n(\alpha)}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_n(\alpha)} \frac{d}{dt}\left(\frac{\partial q_n(\alpha)}{\partial \alpha}\right) = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_n(\alpha)} \frac{\partial q_n(\alpha)}{\partial \alpha}\right] = 0\,.$$

In particular, this holds for $\alpha = 0$. Thus, the quantity

$$\mathcal{N} \equiv \frac{\partial L}{\partial \dot{q}_n} \frac{\partial q_n}{\partial \alpha} = p_n \frac{\partial q_n}{\partial \alpha} \tag{13}$$

is an invariant of the motion. Thus, the symmetry property implies a conservation law. The reverse also holds: every conserved quantity is associated with a symmetry of the dynamical system.

EXAMPLE 4: THE SWINGING SPRING IV. We first illustrate Noether's Theorem in two simple cases: invariance under translation and invariance under rotation. Suppose the Lagrangian is

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

where the potential energy is independent of x and y. The Lagrangian is invariant under the horizontal translation $x \to x + \alpha$. The quantity $\mathcal{N}$ in (13) is just the x-momentum $p_x = m\dot{x}$. Clearly, the y-momentum $p_y = m\dot{y}$ is also a constant of the motion. Thus, translational invariance corresponds to conservation of linear momentum.

Consider now invariance under a rotation:

$$\begin{aligned} x(\alpha) &= +x\cos\alpha + y\sin\alpha \\ y(\alpha) &= -x\sin\alpha + y\cos\alpha \\ z(\alpha) &= z\,. \end{aligned}$$

If the Lagrangian is assumed to be unchanged under this transformation, (13) yields the following conserved quantity:

$$\mathcal{N} = m(y\dot{x} - x\dot{y})\,.$$

This is, of course, the angular momentum about the vertical. Thus, rotational invariance is associated with conservation of angular momentum.

Now recall from Example 3 that the Hamiltonian for envelope amplitudes of the swinging spring is given by $H = 2i\Im\{A_1A_2A_3^*\} = (ABC^* - A^*B^*C)$, where $(A_1, A_2, A_3) = (A, B, C)$ and we take $p = (A, B, C)$ and $q = (A^*, B^*, C^*)$ as canonically conjugate coordinates. The Hamiltonian is unchanged under the following transformations:

1. The phases of A and C are changed by equal amounts;
2. The phases of B and C are changed by equal amounts;
3. The phases of A and B are changed by opposite amounts.

By means of the Legendre transformation, it is clear that the Lagrangian has similar properties of symmetry. Consider the first case, and let $A \to A\exp(i\alpha)$ and $C \to C\exp(i\alpha)$. Using (13), we find that the following quantity is conserved:

$$\mathcal{N} = p_n \frac{\partial q_n}{\partial\alpha}\bigg|_{\alpha=0} = A(-iA^*) + C(-iC^*) = -i(|A|^2 + |C|^2)\,.$$

Thus, $N_1 \equiv |A|^2 + |C|^2$ is a constant of motion. Similarly, $N_2 \equiv |B|^2 + |C|^2$ is invariant (N_1 and N_2 are called the Manley-Rowe quantities). The sum or these is $N = |A|^2 + |B|^2 + 2|C|^2$, which we already noted as an invariant in Example 3 above. The third symmetry yields constancy of the angular momentum quantity, $M = |A|^2 - |B|^2$. We thus see how constants of the motion may be found by scrutiny of the Hamiltonian, without explicit consideration of the equations of motion.

2.5 Casimirs

If the reduction of a Hamiltonian system results in a non-canonical formulation, there are constants of the motion associated with the singular nature of the symplectic operator. In the reduced phase-space, canonical coordinates do not exist. There are conserved quantities corresponding to symmetries of the original system which no longer appear explicitly in the reduced Hamiltonian. These appear as additional constants of motion called *Casimirs*.[2] The reduced Hamiltonian is determined only up to addition of these Casimir functions. The Casimirs of the problem are those functions of z which Poisson-commute with all other state functions or, equivalently, whose z-derivatives are in the kernel of J:

$$\{F, C\} = 0\,, \quad \forall\, F = F(z) \qquad \text{or} \qquad J^{ij}\frac{\partial C}{\partial z^j} = 0\,.$$

[2] According to Marsden and Ratiu (1999), H. B. G. Casimir, a student of Paul Ehrenfest, wrote a brilliant thesis on the quantum mechanics of the rigid body (Casimir, 1931)

They arise due to the singularity of the symplectic operator J because, if J is invertible, its kernel is trivial. For canonical systems there are no non-constant Casimirs. The number of independent Casimirs is the co-rank of the operator J. Through Noether's Theorem, continuous symmetries of the Hamiltonian are associated with invariants of the motion. Casimirs are also constants of the motion, since $dC/dt = \{C, H\} = 0$, but they are determined by the degenerate structure of J, not by the Hamiltonian H. The dynamics are not altered by addition of an arbitrary linear combination of the Casimir functions to the Hamiltonian.

EXAMPLE 5: THE SWINGING SPRING V. For the special case where the Hamiltonian takes the value zero, the system (12) reduces to three *real* equations for $X = |A_1|$, $Y = |A_2|$ and $Z = |A_3|$:

$$\dot{X} = -YZ\,, \quad \dot{Y} = -ZX\,, \quad \dot{Z} = +XY\,. \tag{14}$$

If we define the coordinates as $z^i = (X, Y, Z)^{\mathrm{T}}$ and the Hamiltonian to be $H = \frac{1}{2}(X^2 + Y^2 + 2Z^2)$, this system can be written as a non-canonical Hamiltonian system

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j}$$

where the symplectic matrix takes the (non-unique) form

$$J = J^{ij} = \begin{pmatrix} 0 & -2kZ & -(\frac{1}{2} - k)Y \\ 2kZ & 0 & -(\frac{1}{2} + k)X \\ (\frac{1}{2} - k)Y & (\frac{1}{2} + k)X & 0 \end{pmatrix}.$$

Here k is arbitrary; J takes its simplest forms for $k \in \big\{-\frac{1}{2}, 0, +\frac{1}{2}\big\}$. We note that $|J| = 0$, so the system is non-invertible. In any case, the system is obviously non-canonical, since the order of z is odd. It is straightforward to show that J satisfies the Jacobi identity

$$\varepsilon_{ijk} J^{im} \frac{\partial J^{jk}}{\partial z^m} = 0\,.$$

We define $C(k) = 2kH + M$ where $M = X^2 - Y^2$ is the angular momentum of the spring. Clearly $\partial C(k)/\partial z^j = 4\big((k + \frac{1}{2})X, (k - \frac{1}{2})Y, 2kZ\big)^{\mathrm{T}}$ and so

$$J^{ij}\frac{\partial C(k)}{\partial z^j} = 0\,.$$

Thus, $C(k)$ is a Casimir of the system. Since k is arbitrary, we may choose $k = 0$. Then $M = C(0)$ is a Casimir. Obviously, so is any function $f(M)$. The dynamics are unaffected by addition of $f(M)$ to the Hamiltonian. Thus, for example, we may consider

$$H' = H + \tfrac{1}{2}M = X^2 + Z^2\,,$$

to be the Hamiltonian. This is simpler than the original form, since Y no longer appears in it.

Equations (14) are equivalent to Euler's equations for the rotation of a free rigid body rotating about its center of gravity (see Lynch, 2002c for discussion). For a generalized Hamiltonian formulation of Euler's equations, see Shepherd, 1990.

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3 Continuous Fluid Systems

3.1 Lagrangian and Eulerian Descriptions

In the Lagrangian description, each fluid particle is assigned a label $\mathbf{a} = (a, b, c)$. For example, the labels may be defined in terms of the positions of the particles at the initial time. The label of an individual particle is fixed for all time and travels with the flow. The independent variables are (a,b,c,τ), where we denote time by τ so that $\partial/\partial\tau$ means that (a,b,c) are held fixed. The dependent variables are the position coordinates

$$x(a,b,c,\tau)\,,\quad y(a,b,c,\tau)\,,\quad z(a,b,c,\tau) \tag{15}$$

of the particles as functions of their labels and the time. We assume this transformation is invertible at every time so that (a,b,c) can be obtained in terms of (x,y,z).

The derivatives in $\mathbf{a}$-space and $\mathbf{x}$-space are related by the chain rule

$$\frac{\partial F}{\partial\tau} = \frac{\partial F}{\partial t}\frac{\partial t}{\partial\tau} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial\tau} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial\tau} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial\tau} \tag{16}$$

($\partial/\partial t$ means that (x,y,z) are held constant). But the velocity of a particle is given by the substantive derivative

$$\mathbf{v} = (u,v,w) = \left(\frac{\partial x}{\partial\tau}, \frac{\partial y}{\partial\tau}, \frac{\partial z}{\partial\tau}\right)$$

so (16) can be written

$$\frac{\partial F}{\partial\tau} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}}F$$

which is the usual expression for the Lagrangian time derivative, normally written dF/dt, following the flow.

It is convenient to assign the particle labels in such a way that

$$dm = da\,db\,dc = dV_{\mathbf{a}}\,,$$

where dm is the mass of an infinitesimal volume $dV_{\mathbf{a}}$ in $\mathbf{a}$-space. But

$$dm = \rho\,dx\,dy\,dz = \rho\,dV_{\mathbf{x}}$$

where $dV_{\mathbf{x}}$ is the volume in $\mathbf{x}$-space. The volume expressions are relatated by the Jacobian of the transformation (15) so we have

$$\rho = \frac{\partial(a,b,c)}{\partial(x,y,z)} \equiv \frac{\partial\mathbf{a}}{\partial\mathbf{x}}\,.$$

The specific volume is the inverse of the density,

$$\alpha = \frac{\partial(x,y,z)}{\partial(a,b,c)} \equiv \frac{\partial\mathbf{x}}{\partial\mathbf{a}}\,.$$

and the substantive derivative of this leads to

$$\frac{\partial\alpha}{\partial\tau} = \alpha\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

(see Salmon, 1998, p. 6). We can rewrite this in a more familiar form

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0\,,$$

so the usual continuity equation results from the assumption that fixed $\mathbf{a}$-space volumes have fixed mass.

3.2 Transition from Discrete to Continuous

The momentum equation will be derived from Hamilton's principle (following Salmon, 1998). We will deduce the Lagrangian formulation for fluid flow by transition from a discrete to a continuous system. The Lagrangian for a system of N discrete particles can be written

$$L(\mathbf{x}_n, \dot{\mathbf{x}}_n) = \sum_{n=1}^{N} \tfrac{1}{2} m_n \frac{d\mathbf{x}_n}{dt} \cdot \frac{d\mathbf{x}_n}{dt} - V(\mathbf{x}) \tag{17}$$

where $V(\mathbf{x})$ is the potential function. Hamilton's principle states that the variation

$$\delta \int_{t_1}^{t_2} L(\mathbf{x}_n, \dot{\mathbf{x}}_n)\, dt$$

should be zero for arbitrary variations $\delta\mathbf{x}_n$ that vanish at the end times. This leads to Newton's law

$$\frac{d}{dt}(m_n \dot{\mathbf{x}}_n) = -\frac{\partial V}{\partial \mathbf{x}_n}\,.$$

By allowing N to increase without limit and the distances between particles to become arbitrarily small, we can represent a continuous fluid by a Lagrangian density of the form (17) where now $\mathbf{x}$ and $\dot{\mathbf{x}}$ correspond to the positions and velocities of labeled fluid particles and the indices n represent the particle labels. We replace $\sum m_n$ by $\int dadbdc$. The kinetic energy becomes

$$T = \iiint \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} \right\} dadbdc$$

The potential energy comprises the internal energy $E = E(\alpha, S)$ which is a function of specific volume α and specific entropy S, and the energy due to external forces such as gravity, represented by a potential function $\Phi(\mathbf{x})$ which is a function of position. Thus

$$V = \iiint \{ E(\alpha, S) + \Phi(\mathbf{x}) \}\, dadbdc\,.$$

Then Hamilton's principle requires that the action

$$\int_{t_1}^{t_2} dt \left\{ \iiint \left[\tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}}, S \right) - \Phi(\mathbf{x}) \right] dadbdc \right\} \tag{18}$$

be stationary for arbitrary variations $\delta\mathbf{x}(a, b, c)$ in the locations of the fluid particles.

3.3 The Lagrangian Momentum Equation

The variations $\delta\mathbf{x}$ must have no normal components at rigid boundaries and must vanish at the end times. For adiabatic motion, the entropy S depends only on $\mathbf{a}$ and not on τ. A straightforward calculation then yields the momentum equation

$$\frac{\partial^2 \mathbf{x}}{\partial \tau^2} = -\alpha \nabla p - \nabla \Phi$$

where $p = -\partial E/\partial\alpha = \rho^2\partial E/\partial\rho$ is the pressure (which must vanish at free boundaries). Noting that $\mathbf{v} = \partial\mathbf{x}/\partial\tau$ is the fluid velocity, we can write the momentum equation in a more familiar form:

$$\frac{d\mathbf{v}}{dt} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0\,.$$

The Lagrangian density (the integrand of (18)) does not depend explicitly on the time τ. This symmetry is associated, through Noether's theorem, with energy conservation

$$\frac{d}{dt}\iiint \left\{\tfrac{1}{2}\frac{\partial\mathbf{x}}{\partial\tau}\cdot\frac{\partial\mathbf{x}}{\partial\tau} + E + \Phi\right\} da\,db\,dc = 0\,.$$

There is another, less obvious, symmetry property of (18), corresponding to particle relabeling which leaves the density and entropy unchanged. The conservation principle associated with this particle relabeling symmetry is that of potential vorticity

$$\frac{d}{dt}\left[\frac{\nabla\times\mathbf{v}\cdot\nabla S}{\rho}\right] = 0\,.$$

The particle relabeling symmetry is responsible for the existence of a closed Eulerian formulation of fluid mechanics.

3.4 Functional Derivatives

In discrete mechanical systems the dependent variables z^n are functions of the time τ. A function of state is any function $F(z^n)$ whose value is determined once the variables z^n are specified. Its variation with z^n is given in terms of the partial derivatives:

$$\delta F = \frac{\partial F}{\partial z^n}\delta z^n\,.$$

In fluid dynamics, the dependent variables are functions of space as well as time, $z^n = z^n(\mathbf{x},\tau)$. We assume that they belong to some function space $\mathcal{Z}$. We assume also that a real inner product is defined on this space. Typically, it is a spatial integral over the domain:

$$\langle F, G\rangle = \iiint F(z)G(z)\,d\mathbf{a}\,.$$

Instead of functions of state we have *functionals* of state, that is functions of functions, mapping $\mathcal{Z}$ to the real line. We will denote functionals by script letters. In place of partial derivatives we must now consider *functional derivatives* $\delta\mathscr{F}/\delta z^n$ which are defined by

$$\frac{d}{d\epsilon}\mathscr{F}[z^n + \epsilon w]\bigg|_{\epsilon=0} = \left\langle\frac{\delta\mathscr{F}}{\delta z^n}, w\right\rangle\,, \tag{19}$$

for arbitrary functions $w \in \mathcal{Z}$. Writing $\delta z^n = \epsilon w$ this implies

$$\delta\mathscr{F} \equiv \mathscr{F}[z^n + \delta z^n] - \mathscr{F}[z^n] = \iiint \frac{\delta\mathscr{F}}{\delta z^n}\delta z^n d\mathbf{a} + O((\delta z^n)^2)\,. \tag{20}$$

The variations δz^n are assumed to vanish at the domain boundaries. The functional derivative $\delta\mathscr{F}/\delta z^n$ is itself a function in the space $\mathcal{Z}$. To evaluate the functional derivatives of $\mathscr{F}$, we calculate the variation $\delta\mathscr{F}$ arising from arbitrary variations δz^n, and express it in the form (20).

EXAMPLE 6: FUNCTIONAL DERIVATIVES. To clarify the definition of functional derivatives, we consider three examples. First, let $u(x)$ be a function on the unit interval $[0,1]$ and let

$$\mathscr{F}[u] = \int_0^1 F[x, u(x), u_x(x)]\,dx\,.$$

Then, applying a variation δu with $\delta u(0) = \delta u(1) = 0$, we have

$$\delta\mathscr{F} = \int_0^1 \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x\right]\,dx = \int_0^1 \left[\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x}\right)\delta u\right]\,dx$$

where the last expression arises through integration by parts. Thus,

$$\frac{\delta\mathscr{F}}{\delta u} = \frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x}\,.$$

As a second example, let $\mathscr{F}[u] = u(x_0)$ be the functional which evaluates u at x_0. We write it in the form $\mathscr{F}[u] = \int \delta(x - x_0)u(x)dx$. Thus we have

$$\delta\mathscr{F}[u] = \int \delta(x - x_0)\delta u(x)\,dx = \langle\delta(x - x_0), \delta u\rangle$$

which immediately implies

$$\frac{\delta\mathscr{F}}{\delta u} = \delta(x - x_0)\,.$$

For a final example, let us suppose that the energy of a fluid system can be expressed as

$$\mathscr{E} = \iiint \left[\tfrac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + V(\mathbf{x})\right]d\mathbf{a}\,.$$

Since $d\mathbf{a}$ has dimensions of mass, $\mathscr{E}$ has dimensions $\mathrm{ML}^2\mathrm{T}^{-2}$ of energy, as expected. If $\mathbf{x}$ and $\dot{\mathbf{x}}$ are varied, we have

$$\delta\mathscr{E} = \iiint \left[\dot{\mathbf{x}}\cdot\delta\dot{\mathbf{x}} + \frac{\partial V}{\partial\mathbf{x}}\cdot\delta\mathbf{x}\right]d\mathbf{a}\,,$$

so the variational derivatives are given by

$$\frac{\delta\mathscr{E}}{\delta\dot{\mathbf{x}}} = \dot{\mathbf{x}}\,, \qquad \frac{\delta\mathscr{E}}{\delta\mathbf{x}} = \frac{\partial V}{\partial\mathbf{x}}\,.$$

Note the dimensions: $\delta\mathscr{E}/\delta\dot{\mathbf{x}}[=]\mathrm{LT}^{-1}$. However, $\delta\mathscr{E}[=]\mathrm{ML}^2\mathrm{T}^{-2}$ and $\delta\dot{\mathbf{x}}[=]\mathrm{LT}^{-1}$, so the ratio of these dimensions is MLT^{-1}. Thus, the functional derivative does *not* have the dimensions which its symbolic form would suggest.[3]

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[3] This is a serious defect of the notation. However, the convention is now firmly established.

3.5 The Canonical Equations for a Fluid

Let us suppose that the Lagrangian is given in the form

$$\mathscr{L} = \iiint L(\mathbf{x}, \dot{\mathbf{x}})\, d\mathbf{a}\,.$$

Then the variation of $\mathbf{x}$ and $\dot{\mathbf{x}}$ leads to

$$\delta\mathscr{L} = \iiint \left[\frac{\partial L}{\partial \mathbf{x}}\cdot\delta\mathbf{x} + \frac{\partial L}{\partial \dot{\mathbf{x}}}\cdot\delta\dot{\mathbf{x}}\right]\, d\mathbf{a}$$

or, in terms of variational derivatives,

$$\delta\mathscr{L} = \iiint \left[\frac{\delta \mathscr{L}}{\delta \mathbf{x}}\cdot\delta\mathbf{x} + \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}}\cdot\delta\dot{\mathbf{x}}\right]\, d\mathbf{a}\,.$$

Now Hamilton's principle requires that the action $\mathscr{S} = \int \mathscr{L}\, dt$ be stationary:

$$\delta\mathscr{S} = \delta \int \mathscr{L}\, dt = \int \left\{ \iiint \left[\frac{\delta \mathscr{L}}{\delta \mathbf{x}}\cdot\delta\mathbf{x} + \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}}\cdot\delta\dot{\mathbf{x}}\right]\, d\mathbf{a}\right\} dt = 0$$

Integrating by parts and assuming $\delta\mathbf{x}$ vanishes at the end times, we get the Lagrangian equations for a fluid

$$\frac{d}{dt}\frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}} - \frac{\delta \mathscr{L}}{\delta \mathbf{x}} = 0\,.$$

The generalized momentum $\mathbf{p}$ corresponding to (3) is

$$\mathbf{p} \equiv \mathbf{u} = \frac{\delta \mathscr{L}}{\delta \dot{\mathbf{x}}}\,. \tag{21}$$

By analogy with (4), the Hamiltonian $\mathscr{H}$ is defined by the Legendre transformation

$$\mathscr{H} = \iiint \mathbf{u}\cdot\dot{\mathbf{x}}\, d\mathbf{a} - \mathscr{L}\,,$$

with $\dot{\mathbf{x}}$ expressed in terms of $\mathbf{x}$ and $\mathbf{u}$ using (21). If we now consider Hamilton's principle in the form

$$\delta \int \left\{ \iiint \mathbf{u}\cdot\dot{\mathbf{x}}\, d\mathbf{a} - \mathscr{H}(\mathbf{x}, \mathbf{u}) \right\} dt = 0$$

for arbitrary variations $\delta\mathbf{x}$ and $\delta\mathbf{u}$ in the positions and velocities of labeled fluid particles, the canonical equations emerge in the form

$$\dot{\mathbf{x}} = \frac{\delta \mathscr{H}}{\delta \mathbf{u}}\,, \qquad \dot{\mathbf{u}} = -\frac{\delta \mathscr{H}}{\delta \mathbf{x}}\,.$$

3.6 Conserved Quantities, Noether's Theorem and Casimir Functionals

To avail of the extraordinary simplicity of the Eulerian description of fluid systems, we are led to consider generalized Hamiltonian systems with non-canonical form, capable of representing the Eulerian framework. The simplicity of the Eulerian description is a result of the symmetry property of the Hamiltonian, which remains unchanged under a re-labeling of the fluid particles having the same density and entropy. This motivates the transformation to Eulerian variables but, in the Hamiltonian context, it results in a non-canonical formulation. The transformation is a projection to a reduced set of variables. In the reduced phase-space, canonical coordinates do not exist.

For non-canonical systems, there are invariants of the motion arising from two sources. First, as for canonical systems, symmetries of the Hamiltonian are associated with constants of motion through Noether's theorem. Let z be the set of dependent field variables. If $\mathscr{H}$ and J are invariant under arbitrary translation of a particular coordinate x^k, and $\mathscr{M}$ is a functional satisfying

$$J\frac{\delta\mathscr{M}}{\delta z} = -\frac{\partial z}{\partial x^k}\,, \tag{22}$$

then $\mathscr{M}$ is a constant of the motion: $d\mathscr{M}/dt = 0$.

Additional conserved quantities, the Casimirs, appear in the non-canonical formulation. They are solutions of the equation

$$J\frac{\partial\mathscr{C}}{\partial z} = 0\,.$$

Since this is a homogeneous version of (22), solutions of that equation are determined only up to addition of a Casimir. Thus, for example, a Casimir may be added to the Hamiltonian without any effect on the dynamics.

EXAMPLE 7: GUESSING THE HAMILTONIAN FORMULATION As an example of how the Hamiltonian formulation of a fluid dynamical system may be derived in a heuristic manner, consider non-divergent two-dimensional flow in a non-rotating coordinate system. The equations of motion are

$$\frac{d\mathbf{V}}{dt} + \frac{1}{\rho_0}\nabla p = 0\,, \qquad \nabla\cdot\mathbf{V} = 0\,,$$

where $\mathbf{V} = (u, v)$ is the non-divergent velocity, ρ_0 is the constant density and p the pressure. Assuming a simply-connected domain with impermeable boundary, the conserved energy is

$$E = \iint \tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\,dxdy\,.$$

The velocity may be expressed in terms of a stream function ψ, such that $(u, v) = (-\psi_y, +\psi_x)$. The vorticity $\zeta = \nabla^2\psi$ is conserved following the flow:

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + \partial(\psi, \zeta) = 0 \tag{23}$$

where $\partial(\alpha, \beta) = \alpha_x\beta_y - \alpha_y\beta_x$ is the Jacobian.

We now proceed by intuition. Since the vorticity is a fundamental conserved quantity, we choose it as the Hamiltonian variable, $z = \zeta$. The Hamiltonian is guessed to be the energy, written

$$\mathscr{H} = \iint \tfrac{1}{2}\nabla\psi\cdot\nabla\psi\,dxdy\,.$$

Then the variational derivative of $\mathscr{H}$ is easily calculated:

$$\delta\mathscr{H} = \iint \nabla\psi\cdot\nabla\delta\psi\, dxdy = \oint \psi\nabla\delta\psi\cdot\mathbf{n}\, ds - \iint \psi\nabla^2\delta\psi\, dxdy = \iint (-\psi)\delta z\, dxdy\,,$$

so that $\delta\mathscr{H}/\delta z = -\psi$ (we have taken the arbitrary constant boundary value of ψ to be zero). We know $z_t = \partial(z,\psi)$, so we can write (23) in generalized Hamiltonian form

$$\frac{\partial z}{\partial t} = J\frac{\delta\mathscr{H}}{\delta z} \tag{24}$$

where the symplectic operator is given by $J(F) = -\partial(z,F)$. We may alternatively write the system in terms of a Poisson bracket:

$$\frac{\partial\mathscr{F}}{\partial t} = \{\mathscr{F},\mathscr{H}\} = -\iint \frac{\delta\mathscr{F}}{\delta z}\partial\left(z,\frac{\delta\mathscr{H}}{\delta z}\right)\, dxdy\,.$$

The Casimirs of the system are the solutions of

$$J\frac{\delta\mathscr{C}}{\delta z} = \partial\left(z,\frac{\delta\mathscr{C}}{\delta z}\right) = 0\,.$$

Clearly, $J(z) = 0$ and indeed any function $f(z)$ is annihilated by J. Thus,

$$\frac{\delta\mathscr{C}(z)}{\delta z} = f(z) \quad \text{or} \quad \mathscr{C}(z) = \iint\left[\int f(z)\, dz\right] dxdy\,.$$

so that the Casimirs of the system are just integrals of arbitrary functions of the vorticity.

To conclude that (24) is in generalized Hamiltonian form, it is necessary to prove that the symplectic operator J has the required properties. Obviously, it is skew-symmetric in its arguments. The proof that it satisfies the Jacobi identity requires more algebra, which we omit (see, e.g., Swaters (2000) for details).

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4 Reduction from Lagrangian to Eulerian Form

Normally, the Lagrangian (i.e., particle-following) representation of fluid systems leads to a canonical Hamiltonian formulation. This is generally not the case for the Eulerian formulation. One may well wonder how the non-canonical formulation of a general Eulerian fluid-dynamical system may be found. A number of specific formulations have been arrived at by inspired guess-work. However, this is not a satisfactory method in general, Moreover, it requires an explicit demonstration that the posited symplectic operator has the appropriate algebraic properties. The proof that a given J or Poisson bracket satisfies the Jacobi identity can be challenging.

A more satisfactory way of deducing the Hamiltonian formulation of an Eulerian representation is by reduction of the (canonical) Lagrangian formulation. This has at least three advantages. It allows us to proceed in a logical and deductive fashion, which is more appealing that guess-work. It ensures that the algebraic properties of the Lagrangian formulation carry over to the Eulerian representation. It elucidates the origin of the Casimir functions, which are absent in the canonical formulation. These functions arise through symmetries in the Lagrangian form arising from the particle-relabeling symmetry, which is hidden in the Eulerian framework.

In the Lagrangian formulation, the state of a fluid system is specified by means of the position **x(a,**τ**)** and velocity **u(a,**τ**)** of every fluid particle, as functions of the time τ. The particles are labeled by coordinates $\mathbf{a}$, the label of each particle remaining unchanged. Thus, the state of the system at any time is determined by the six Lagrangian fields

$$\{\mathbf{x}(\mathbf{a}), \mathbf{u}(\mathbf{a})\}\,.$$

In the Eulerian formulation, the required fields are the velocity, density and entropy given as functions of position:

$$\{\mathbf{u}(\mathbf{x}), \rho(\mathbf{x}), S(\mathbf{x})\}\,. \tag{25}$$

These are determined uniquely from the Lagrangian fields. However, as there are only five Eulerian fields, they may be represented as points in a reduced phase-space.

For two arbitrary functionals of the Lagrangian state of the fluid, $\mathscr{F}(\mathbf{x}(\mathbf{a}), \mathbf{u}(\mathbf{a}))$ and $\mathscr{G}(\mathbf{x}(\mathbf{a}), \mathbf{u}(\mathbf{a}))$, the Poisson bracket is defined by

$$\{\mathscr{F},\mathscr{G}\}_{\mathrm{L}} = \iiint \left\{ \frac{\delta\mathscr{F}}{\delta\mathbf{x}(\mathbf{a})}\cdot\frac{\delta\mathscr{G}}{\delta\mathbf{u}(\mathbf{a})} - \frac{\delta\mathscr{F}}{\delta\mathbf{u}(\mathbf{a})}\cdot\frac{\delta\mathscr{G}}{\delta\mathbf{x}(\mathbf{a})} \right\} d\mathbf{a} \tag{26}$$

If we consider an arbitrary transformation to new independent variables $\mathbf{y}$ *and* new dependent variables $v_i(\mathbf{y})$, the bracket may be expressed as

$$\{\mathscr{F},\mathscr{G}\} = \iiint\iiint \left\{ \frac{\delta\mathscr{F}}{\delta v_i(\mathbf{y}_1)}\{v_i(\mathbf{y}_1), v_j(\mathbf{y}_2)\}_{\mathrm{L}}\frac{\delta\mathscr{G}}{\delta v_j(\mathbf{y}_2)} \right\} d\mathbf{y}_1 d\mathbf{y}_2 \tag{27}$$

where $\{v_i(\mathbf{y}_1), v_j(\mathbf{y}_2)\}_{\mathrm{L}}$ is calculated using (26).

EXAMPLE 8: THE CHAIN RULE. First, consider a finite-dimensional system, and consider a coordinate transformation $Z^i = Z^i(z)$. The symplectic operator in terms of the new variables is

$$\tilde{J}^{ij} = \frac{\partial Z^i}{\partial z^m} J^{mn} \frac{\partial Z^j}{\partial z^n} \tag{28}$$

It is a second order contravariant tensor.

There is a formal similarity between (28) and (27). The latter is the generalization to continuous fields of the discrete transformation. We consider a change from Lagrangian independent variables $\mathbf{a}$ to Eulerian variables $\mathbf{x}(\mathbf{a})$ and of dependent Lagrangian fields $u_k = \{\mathbf{x}(\mathbf{a}), \mathbf{u}(\mathbf{a})\}$ to Eulerian fields $v_k = \{\mathbf{u}(\mathbf{x}), \rho(\mathbf{x}), S(\mathbf{x})\}$. Then the variational derivative of an arbitrary functional $\mathscr{F}$ with respect to the old variables is given in terms of the new ones by

$$\frac{\delta\mathscr{F}}{\delta u_i} = \iiint \frac{\delta\mathscr{F}}{\delta v_k}\frac{\delta v_k}{\delta u_i}\, d\mathbf{x}\,.$$

This is the chain rule for variational derivatives. If it is applied to the Poisson bracket (26), the transformation rule (27) results.

If the functionals $\mathscr{F}$ and $\mathscr{G}$ depend on $\mathbf{x}(\mathbf{a})$ and $\mathbf{u}(\mathbf{a})$ only through the Eulerian fields (25), we can use (27) to derive a Poisson bracket in terms of the Eulerian fields. The algebra is formidable and we just present the result here; see Morrison and Greene (1980) for the lengthy calculations. The Eulerian bracket is

$$\{\mathscr{F},\mathscr{G}\}_{\mathrm{E}} = -\iiint \Bigg\{ \qquad \left[\frac{\delta\mathscr{F}}{\delta\rho}\nabla\cdot\frac{\delta\mathscr{G}}{\delta\mathbf{u}} + \frac{\delta\mathscr{F}}{\delta\mathbf{u}}\cdot\nabla\frac{\delta\mathscr{G}}{\delta\rho} \right]$$

$$
\begin{aligned}
&+ \quad \left[\frac{\nabla\times\mathbf{u}}{\rho}\cdot\left(\frac{\delta\mathscr{G}}{\delta\mathbf{u}}\times\frac{\delta\mathscr{F}}{\delta\mathbf{u}}\right)\right] \\
&+ \quad \left[\frac{\nabla S}{\rho}\cdot\left(\frac{\delta\mathscr{F}}{\delta S}\frac{\delta\mathscr{G}}{\delta\mathbf{u}}-\frac{\delta\mathscr{G}}{\delta S}\frac{\delta\mathscr{F}}{\delta\mathbf{u}}\right)\right] \quad \Big\} d\mathbf{x}
\end{aligned}
\tag{29}
$$

A more general formulation, applicable to magnetohydrodynamics, and a hierarchy of simplifications of it, are discussed in Morrison (1982).

This Poisson bracket provides a Hamiltonian formulation in terms of the Eulerian variables (25). The evolution of the system is given by

$$\frac{d\mathscr{F}}{dt} = \{\mathscr{F},\mathscr{H}\}_{\mathrm{E}}$$

where the Hamiltonian is

$$\mathscr{H} = \iiint \rho\left\{\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1},S) + \Phi\right\}\, d\mathbf{x}\,.$$

The bracket (29) inherits crucial properties from its Lagrangian forbear (26). It is skew-symmetric and satisfies the Jacobi identity. However, it is singular. If we define the potential vorticity by

$$q = \frac{\nabla\times\mathbf{u}\cdot\nabla S}{\rho}\,,$$

it can be shown that for any functional $\mathscr{C}(q)$ one has

$$\{\mathscr{F},\mathscr{C}\}_{\mathrm{E}} = 0$$

for all functionals $\mathscr{F}$. The singular nature of the Poisson bracket is a consequence of the projective nature of the transformation from Lagrangian to Eulerian coordinates.

EXAMPLE 9: A SIMPLE REDUCTION. To illustrate the algebraic process of reduction of the Poisson bracket in a simple case, we consider the one-dimensional, non-rotating shallow water equations. Assume an incompressible fluid of density ρ; without loss of generality, we may set $\rho = 1$. The Lagrangian coordinates are (a, b, c) but, as there is no y-dependence, b is constant and is ignored. The position and velocity are $x(a, c, \tau)$ and $\dot{x}(a, c, \tau)$, but we assume material columns remain vertical so that $x = x(a, \tau)$, $\dot{x} = \dot{x}(a, \tau)$. Labels are assigned so that

$$da = h\,dx \tag{30}$$

where $h(a, \tau)$ is the depth. A mass element is given by $dm = dadc = dxdz$ which implies $dz = h\,dc$ or, after vertical integration, that c varies from zero at $z = 0$ to unity at $z = h$. Differentiation of (30) yields the continuity equation

$$\frac{\partial h}{\partial\tau} + h\frac{\partial u}{\partial x} = 0\,.$$

The Lagrangian is given by

$$\mathscr{L} = \iint [\tfrac{1}{2}\dot{x}^2 - gz]dadc = \int \tfrac{1}{2}[\dot{x}^2 - gh]da$$

and the resulting Euler-Lagrange equation is

$$\frac{\partial\dot{x}}{\partial\tau} + g\frac{\partial h}{\partial x} = 0\,.$$

The generalized momentum is $p \equiv \delta\mathscr{L}/\delta\dot{x} = \dot{x} \equiv u$, and the Hamiltonian is

$$\mathscr{H} = \int \tfrac{1}{2}[u^2 + gh]da\,.$$

where $h = h(x)$ is shorthand for da/dx.

The Lagrangian independent variables are a and τ and the dependent fields are $x = x(a,\tau)$ and $u = u(a,\tau)$, which are canonical coordinates. The Eulerian independent variables are x and t and the dependent fields are $u = u(x,t)$ and $h = h(x,t)$. The canonical Poisson bracket is defined by (26):

$$\{\mathscr{F},\mathscr{G}\}_{\mathrm{L}} = \int \left\{ \frac{\delta\mathscr{F}}{\delta x(a)}\cdot\frac{\delta\mathscr{G}}{\delta u(a)} - \frac{\delta\mathscr{F}}{\delta u(a)}\cdot\frac{\delta\mathscr{G}}{\delta x(a)} \right\} da\,.$$

To apply (27), we need to evaluate the fundamental brackets such as $\{u(x_1), u(x_2)\}_{\mathrm{L}}$, and this in turn requires knowledge of $\delta u(x_1)/\delta x(a)$ and other similar variational derivatives. We must express $u(x_1)$ and $h(x_1)$ as functionals of $x(a)$ and $u(a)$. We write

$$u(x_1) = \int u(a)\delta(a-a_1)\,da\,,$$

where a_1 is the label corresponding to x_1. Thus

$$\frac{\delta u(x_1)}{\delta x(a)} = 0 \qquad \text{and} \qquad \frac{\delta u(x_1)}{\delta u(a)} = \delta(a-a_1)\,.$$

Similarly, expressing $h(x_1)$ as

$$h(x_1) = \int h(x)\delta(x-x_1)\,dx = \int \delta(x(a)-x(a_1))\,da\,,$$

we find that

$$\frac{\delta h(x_1)}{\delta x(a)} = \frac{\partial}{\partial x}\delta(x-x_1) \qquad \text{and} \qquad \frac{\delta h(x_1)}{\delta u(a)} = 0\,.$$

These expressions enable us to evaluate the fundamental brackets, giving

$$\{u(x_1), u(x_2)\}_{\mathrm{L}} = 0\,, \quad \{h(x_1), h(x_2)\}_{\mathrm{L}} = 0\,, \quad \{u(x_1), h(x_2)\}_{\mathrm{L}} = -\frac{\partial}{\partial x_1}\delta(x_1-x_2)\,.$$

Then the Eulerian bracket is given by

$$\{\mathscr{F},\mathscr{G}\}_{\mathrm{E}} = \iint \left\{ \frac{\delta\mathscr{F}}{\delta u(x_1)}\{u(x_1),h(x_2)\}_{\mathrm{L}}\frac{\delta\mathscr{G}}{\delta h(x_2)} + \frac{\delta\mathscr{F}}{\delta h(x_1)}\{h(x_1),u(x_2)\}_{\mathrm{L}}\frac{\delta\mathscr{G}}{\delta u(x_2)} \right\} dx_1 dx_2\,,$$

and, after insertion of the expressions for the Lagrangian brackets on the right, we arrive at

$$\{\mathscr{F},\mathscr{G}\}_{\mathrm{E}} = \int \left\{ \frac{\partial}{\partial x}\left(\frac{\delta\mathscr{F}}{\delta u}\right)\frac{\delta\mathscr{G}}{\delta h} - \frac{\delta\mathscr{F}}{\delta h}\frac{\partial}{\partial x}\left(\frac{\delta\mathscr{G}}{\delta u}\right) \right\} dx\,. \tag{31}$$

The Hamiltonian in the Eulerian framework is

$$\mathscr{H} = \int \tfrac{1}{2}[u^2 + gh]hdx$$

which implies the functional derivatives $\delta\mathscr{H}/\delta u = hu$ and $\delta\mathscr{H}/\delta h = \frac{1}{2}u^2 + gh$. The Poisson bracket (31) enables us to identify the symplectic operator J through the relationship $\{\mathscr{F},\mathscr{G}\} = \langle\mathscr{F}, J\mathscr{G}\rangle$. After integration by parts, assuming periodic boundary conditions, we see that

$$J = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}\,. \tag{32}$$

Defining $z = (u, h)^{\mathrm{T}}$, the generalized Hamiltonian system of equations

$$\frac{\partial z}{\partial t} = J\frac{\delta\mathscr{H}}{\delta z} \tag{33}$$

then yields the usual Eulerian form of the equations of motion:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\tfrac{1}{2}u^2 + gh) = 0\,, \qquad \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0\,.$$

The Eulerian Poisson bracket (31) automatically satisfies the Jacobi identity, as it is derived from the canonical form by means of a covariant transformation

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5 Shallow Water Equations

We consider the motion of a shallow layer of incompressible fluid above a flat surface in a rotating coordinate system. The height of the free surface is h. It is dynamically consistent to assume material columns remain vertical, and that each vertical column is identified by two Lagrangian labels, a and b. Then the position of a column is

$$x = x(a, b, \tau)\,, \qquad y = y(a, b, \tau)\,.$$

We require these labels to be assigned in such a way that

$$h = \frac{\partial(a, b)}{\partial(x, y)}\,. \tag{34}$$

Then the third, or vertical, label c must be chosen so that

$$\rho_0 = \frac{\partial(a, b, c)}{\partial(x, y, z)} = \frac{\partial(a, b)}{\partial(x, y)}\frac{\partial c}{\partial z} = h\frac{\partial c}{\partial z}\,. \tag{35}$$

This can be integrated immediately to give

$$z = \frac{hc}{\rho_0}\,,$$

so that c varies from zero at $z = 0$ to ρ_0 at the free surface. The continuity equation follows through differentiation of (34):

$$\frac{\partial h}{\partial \tau} + h\nabla\cdot\mathbf{v} = 0\,, \tag{36}$$

where $\mathbf{v} = (u, v) = (\partial x/\partial\tau, \partial y/\partial\tau)$.

EXAMPLE 10: ROTATING COORDINATES. A particle in a potential field $V(\mathbf{X})$ has Lagrangian

$$L = \tfrac{1}{2}\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} - V(\mathbf{X})\,.$$

Let $\mathbf{x}$ be the coordinates in a frame rotating with constant angular velocity $\mathbf{\Omega}$. For simplicity, we assume that the potential energy is unaffected by rotation, so that $V(\mathbf{x}) = V(\mathbf{X})$. The velocities in the two frames are related by $\dot{\mathbf{X}} = \dot{\mathbf{x}} + \mathbf{\Omega} \times \mathbf{x}$, so the Lagrangian in the rotating frame is

$$L = \tfrac{1}{2}\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \mathbf{\Omega} \times \mathbf{x} \cdot \dot{\mathbf{x}} + \tfrac{1}{2}(\mathbf{\Omega} \times \mathbf{x})^2 - V(\mathbf{x})\,. \tag{37}$$

Using (2), the equations of motion may be written immediately:

$$\ddot{\mathbf{x}} + \mathbf{2\Omega} \times \dot{\mathbf{x}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) + \frac{\partial V}{\partial \mathbf{x}} = 0\,. \tag{38}$$

Here we have used the vector identity $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$, which leads to

$$\frac{\partial}{\partial \mathbf{x}}[\tfrac{1}{2}(\mathbf{\Omega} \times \mathbf{x})^2] = \Omega^2\mathbf{x} - (\mathbf{\Omega} \cdot \mathbf{x})\mathbf{\Omega} = -\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x})\,.$$

The second term in (38) is the Coriolis term. The third term is the centrifugal force, which is often combined with V to give an effective potential; we assume this done.

The Hamiltonian is defined by introducing the generalized momentum

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}} + \mathbf{\Omega} \times \mathbf{x}\,.$$

We note that $\mathbf{p}$ is *not* the usual momentum in the rotating frame, but the absolute momentum. The Legendre transformation now yields

$$H = \mathbf{p} \cdot \dot{\mathbf{x}} - L = \tfrac{1}{2}\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + V(\mathbf{x})\,.$$

Mirabile visu, the rotation $\mathbf{\Omega}$ appears to have disappeared. But no! We must express H as a function of the canonical variables $\mathbf{p}$ and $\mathbf{x}$:

$$H = \tfrac{1}{2}(\mathbf{p} - \mathbf{\Omega} \times \mathbf{x}) \cdot (\mathbf{p} - \mathbf{\Omega} \times \mathbf{x}) + V(\mathbf{x}) \tag{39}$$

so its dependence on rotation is now explicit. The canonical equations (5) now yield $\dot{\mathbf{x}} = \partial H/\partial \mathbf{p}$, together with (38).

5.1 Lagrangian Equations in a Rotating Frame

To allow for the Coriolis effect, it is necessary to include additional terms in the Lagrangian. We introduce the notional *world-wind*, $\mathbf{W}$, which is the velocity, measured in an absolute frame, of an atmosphere in solid rotation with the earth:

$$\mathbf{W} = (\mathrm{U}, \mathrm{V}, 0) = \mathbf{\Omega} \times \mathbf{x}\,.$$

On the sphere, the world-wind is, in geographic coordinates, $\mathbf{W} = (\Omega a \cos\phi, 0, 0)$. It has vorticity equal to the planetary vorticity f:

$$\mathbf{k} \cdot \nabla \times \mathbf{W} = \frac{1}{a\cos\phi}\left[\frac{\partial \mathrm{V}}{\partial \lambda} - \frac{\partial}{\partial \phi}(\mathrm{U} \cdot \cos\phi)\right] = 2\Omega \sin\phi = f\,.$$

For an f-plane, with $\mathbf{\Omega} = \Omega\mathbf{k}$, we have $\mathbf{W} = (-\Omega y, \Omega x, 0)$. In any case, the relative and absolute vorticity are given by

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{v} \qquad \text{and} \qquad \zeta + f = \mathbf{k} \cdot \nabla \times (\mathbf{v} + \mathbf{W})\,.$$

The total energy of the fluid is given, in Lagrangian variables, by

$$\mathscr{E} = \iiint \left[\tfrac{1}{2}\frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} + \mathbf{W}\cdot\frac{\partial \mathbf{x}}{\partial \tau} + \tfrac{1}{2}\mathbf{W}\cdot\mathbf{W} + E\left(\frac{\partial \mathbf{x}}{\partial \mathbf{a}}, S\right) + \Phi(\mathbf{x}) \right] da\,db\,dc\,. \tag{40}$$

The gravitational potential is $\Phi = gz$. For incompressible, adiabatic flow, the internal energy $E(\alpha, S)$ may be omitted, as it is unaffected by variations in the positions $\delta\mathbf{x}$ and plays no role in the dynamics. The constant density is given by (35). We neglect the small term in the kinetic energy involving the vertical velocity. The vertical integration may be carried out explicitly, yielding

$$\mathscr{E} = \iint \tfrac{1}{2}\rho_0 \left\{ \frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} + 2\mathbf{W}\cdot\frac{\partial \mathbf{x}}{\partial \tau} + \mathbf{W}\cdot\mathbf{W} + gh \right\} da\,db\,. \tag{41}$$

For simplicity, and without loss of generality, we will assume $\rho_0 = 1$. In the Lagrangian framework, the variable h is defined by (34). The Lagrangian, required to derive the equations from the least action principle, is

$$\mathscr{L} = \iint \tfrac{1}{2} \left\{ \frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} + 2\mathbf{W}\cdot\frac{\partial \mathbf{x}}{\partial \tau} + \mathbf{W}\cdot\mathbf{W} - gh \right\} da\,db\,. \tag{42}$$

Applying the least action principle, we arrive at the equations of motion

$$\frac{\partial^2 \mathbf{x}}{\partial \tau^2} + 2\boldsymbol{\Omega}\times\frac{\partial \mathbf{x}}{\partial \tau} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) + g\nabla h = 0\,. \tag{43}$$

Here, the spatial variation of $\mathbf{W}$ has been carefully allowed for, and we have used the following relationship, which is proved without difficulty:

$$\iint F\delta h\, da\,db = \iint \frac{1}{h}\nabla F h^2\cdot\delta\mathbf{x}\, da\,db \tag{44}$$

(Salmon (1998), p. 317). In the *traditional approximation*, only the vertical component of $\boldsymbol{\Omega}$ is retained in the Coriolis term. Then (43), together with the continuity equation (36), may be written

$$\frac{\partial^2 x}{\partial \tau^2} - f\frac{\partial y}{\partial \tau} + g\frac{\partial h}{\partial x} = 0 \tag{45}$$

$$\frac{\partial^2 y}{\partial \tau^2} + f\frac{\partial x}{\partial \tau} + g\frac{\partial h}{\partial y} = 0 \tag{46}$$

$$\frac{\partial h}{\partial \tau} + h\nabla\cdot\mathbf{v} = 0\,. \tag{47}$$

(Since the centrifugal term depends only on position, it has been absorbed in the gravitational potential by adjusting the value of g). This is the shallow water system of equations in Lagrangian form.

5.2 Hamiltonian Form of the Equations

The generalized momentum is defined as

$$\mathbf{p} \equiv \mathbf{u} = \frac{\delta L}{\delta \dot{\mathbf{x}}} = \dot{\mathbf{x}} + \mathbf{W}\,.$$

Thus, $\mathbf{p}$ is the absolute momentum. The Hamiltonian is now obtained via the Legendre transformation

$$\mathscr{H} = \iint \mathbf{u}\cdot\dot{\mathbf{x}}\, da db - \mathscr{L} = \iint \tfrac{1}{2}\left[\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} - \mathbf{W}\cdot\mathbf{W} + gh\right] da db\,,$$

This must be expressed in terms of the canonical variables $\mathbf{u}$ and $\mathbf{x}$:

$$\mathscr{H} = \iint \left[\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{W} + \tfrac{1}{2}gh\right] da db\,,$$

so that the world-wind appears explicitly in $\mathscr{H}$. Taking a variation of this we get

$$\delta\mathscr{H} = \iint \Big[(\mathbf{u}-\mathbf{W})\cdot\delta\mathbf{u} + \boldsymbol{\Omega}\times\mathbf{u}\cdot\delta\mathbf{x} + g\nabla h\cdot\delta\mathbf{x}\Big] da db\,.$$

Thus, the variational derivatives of $\mathscr{H}$ may be obtained:

$$\frac{\delta\mathscr{H}}{\delta\mathbf{x}} = \boldsymbol{\Omega}\times\mathbf{u} + g\nabla h\,, \qquad \frac{\delta\mathscr{H}}{\delta\mathbf{u}} = \mathbf{u} - \mathbf{W}\,. \tag{48}$$

Defining $z = (\mathbf{x},\mathbf{u})^{\mathrm{T}}$, the equations of motion may be written as

$$\frac{\partial z}{\partial \tau} = J\frac{\delta\mathscr{H}}{\delta z}\,,$$

where the symplectic operator J is in the canonical form (9). The Poisson bracket also takes the canonical form

$$\{\mathscr{F},\mathscr{G}\}_{\mathrm{L}} = \iint \left\{\frac{\delta\mathscr{F}}{\delta\mathbf{x}(\mathbf{a})}\cdot\frac{\delta\mathscr{G}}{\delta\mathbf{u}(\mathbf{a})} - \frac{\delta\mathscr{F}}{\delta\mathbf{u}(\mathbf{a})}\cdot\frac{\delta\mathscr{G}}{\delta\mathbf{x}(\mathbf{a})}\right\} d\mathbf{a}\,. \tag{49}$$

5.3 Eulerian Form of the Equations

We could derive the Eulerian form of the Poisson bracket by transformation of (49), following the method of Example 9, but will deduce it by more intuitive reasoning (this section borrows heavily from Shepherd, 1990). The Eulerian independent variables are $\mathbf{x} = (x,y)$ and time t. The dependent variables are $\mathbf{v} = (u,v)$ and h, and we will explore whether $z = (u,v,h)^{\mathrm{T}}$ are suitable coordinates. We will *guess* that an appropriate form for the Hamiltonian is

$$\mathscr{H} = \iint \tfrac{1}{2}h\Big\{\mathbf{v}\cdot\mathbf{v} + gh\Big\}\, dx dy\,. \tag{50}$$

The factor h comes from the transformation $\int(\ )da db = \int(\ )h\,dxdy$. We consider the variation of $\mathscr{H}$ resulting from variations of the variables $\mathbf{v}$ and h, and find that

$$\frac{\delta\mathscr{H}}{\delta h} = \tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + gh\,, \qquad \frac{\delta\mathscr{H}}{\delta\mathbf{v}} = h\mathbf{v}\,.$$

If the advection is transformed using the vector identity

$$\mathbf{v}\cdot\nabla\mathbf{v} = \nabla\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v} + \zeta\mathbf{k}\times\mathbf{v}$$

where $\zeta = \mathbf{k}\cdot\nabla\times\mathbf{v}$ is the vorticity, the equations (45) to (47) can be written

$$\begin{aligned}
\frac{\partial u}{\partial t} &= +(\zeta+f)v - \frac{\partial}{\partial x}(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+gh)\\
\frac{\partial v}{\partial t} &= -(\zeta+f)u - \frac{\partial}{\partial y}(\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+gh)\\
\frac{\partial h}{\partial t} &= -\frac{\partial}{\partial x}(hu) - \frac{\partial}{\partial y}(hv)
\end{aligned}$$

(once again, the centrifugal term is absorbed in g). Defining the potential vorticity

$$q = \frac{\mathbf{k}\cdot\nabla\times\mathbf{v}+f}{h} = \frac{\zeta+f}{h}\,,$$

the system may be written in generalized Hamiltonian form

$$\frac{\partial z}{\partial t} = J\frac{\delta\mathscr{H}}{\delta z} \tag{51}$$

where $z = (u, v, h)^{\mathrm{T}}$ and the symplectic operator J is given by

$$J = \begin{pmatrix} 0 & q & -\partial_x \\ -q & 0 & -\partial_y \\ -\partial_x & -\partial_y & 0 \end{pmatrix}\,. \tag{52}$$

A Poisson bracket is associated with J through the relationship $\{\mathscr{F},\mathscr{G}\} = \langle\mathscr{F}, J\mathscr{G}\rangle$. Although the matrix operator J in (52) is not skew-symmetric, the appropriate symmetry of the Poisson bracket appears after integrations by parts. The bracket may be written

$$\{\mathscr{F},\mathscr{G}\} = \iint \left\{ q\mathbf{k}\cdot\left(\frac{\delta\mathscr{F}}{\delta\mathbf{v}}\times\frac{\delta\mathscr{G}}{\delta\mathbf{v}}\right) + \left(\nabla\cdot\frac{\delta\mathscr{F}}{\delta\mathbf{v}}\right)\frac{\delta\mathscr{G}}{\delta h} - \frac{\delta\mathscr{F}}{\delta h}\left(\nabla\cdot\frac{\delta\mathscr{G}}{\delta\mathbf{v}}\right)\right\} dxdy \tag{53}$$

which is clearly a skew-symmetric bilinear operator. It can be shown that this Poisson bracket satisfies the Jacobi identity.

6 Balanced Equations

One of the attractions of the Hamiltonian formalism is that it guarantees that conservation properties of an exact system are maintained in approximations thereto if appropriate symmetries of the exact system are preserved in the approximation process. Thus, the time-invariance of the approximate Hamiltonian will ensure an appropriate energy principle, and the particle-relabeling symmetry will mean that potential energy for the approximate system is conserved. The Hamiltonian perspective can also serve as a guide to the choice of new dependent and independent variables in which the approximate equations take their simplest form.

6.1 Non-divergence Constraint in Lagrange Variables

As a simple example of further approximation to the shallow water equations, we impose a constraint that the horizontal flow be non-divergent:

$$\nabla \cdot \dot{\mathbf{x}} = 0\,.$$

This constraint is imposed through the procedure of incorporating an additional term with Lagrange multiplier in the integrand in Hamilton's Principle. We require that the variation of the action, modified by the constraint, be stationary:

$$\delta \int \left\{ \mathscr{L} + \iint \lambda(\nabla \cdot \dot{\mathbf{x}})\, da db \right\} d\tau = 0$$

where $\lambda = \lambda(a, b, \tau)$ is the Lagrange multiplier, to be found, and $\mathscr{L}$ is the Lagrangian of the shallow water equations, given by (42). The variations $\delta\mathbf{x}$ (taken for fixed particle labels and times) are arbitrary but vanish at the end times and the domain boundaries. The solution for unconstrained motion was derived in §5.1 and is given by (43). Additional terms arise through the constraint:

$$\delta \int d\tau \left\{ \iint \lambda(\nabla \cdot \dot{\mathbf{x}}) da db \right\} = \int d\tau \left\{ \iint \left[(\nabla \cdot \dot{\mathbf{x}})\delta\lambda + \lambda(\nabla \cdot \delta\dot{\mathbf{x}}) \right] da db \right\}\,.$$

The vanishing of the coefficient of $\delta\lambda$ ensures that the constraint is satisfied at all locations and all times. The second term is, after some manipulation,

$$\int d\tau \left\{ \iint \lambda(\nabla \cdot \delta\dot{\mathbf{x}}) da db \right\} = \int d\tau \left\{ \iint \left[\frac{1}{h}\frac{\partial}{\partial\tau}\nabla(h\lambda) \right] \cdot \delta\mathbf{x}\, da db \right\}$$

so that (43) is augmented by an additional term:

$$\frac{\partial^2\mathbf{x}}{\partial\tau^2} + 2\boldsymbol{\Omega}\times\frac{\partial\mathbf{x}}{\partial\tau} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) + g\nabla h + \frac{1}{h}\frac{\partial}{\partial\tau}\nabla(h\lambda) = 0\,. \tag{54}$$

The continuity equation (36) reduces in this case to the consistency condition

$$\frac{\partial h}{\partial\tau} = 0\,.$$

This is physically reasonable: an individual fluid column cannot vary in height as, without any influx or outflow, there is no mechanism for supplying or removing fluid. Thus, flow must be along contours of constant height h.

How do we calculate λ? The momentum equation may be written

$$\frac{d}{dt}\frac{1}{h}\nabla(h\lambda) + \frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega}\times\mathbf{u} + g\nabla h = 0\,. \tag{55}$$

where $\mathbf{u} = d\mathbf{x}/dt$ and the centrifugal term is included in g. We take the divergence of this and use the fact that $\nabla\cdot\mathbf{u} = 0$ to obtain a prognostic equation for λ. We note the relationship, valid for any vector:

$$\nabla\cdot\frac{d\mathbf{C}}{dt} = \frac{d}{dt}(\nabla\cdot\mathbf{C}) + (\nabla\cdot\mathbf{C})^2 - 2J(C_1, C_2)\,.$$

Defining the modified gradient and Laplacian operators

$$\tilde{\nabla}\lambda \equiv \frac{1}{h}\nabla(h\lambda) \qquad \text{and} \qquad \tilde{\nabla}^2\lambda \equiv \nabla\cdot\tilde{\nabla}\lambda$$

and using the above relationship, the following equation for λ results:

$$\frac{d}{dt}(\tilde{\nabla}^2\lambda) + (\tilde{\nabla}^2\lambda)^2 - 2J((\tilde{\nabla}\lambda)_1, (\tilde{\nabla}\lambda)_2) = -\Big[g\nabla^2 h + f\zeta - \beta u - 2J(u,v)\Big]\,.$$

This can be used as a predictive equation for $\tilde{\nabla}^2\lambda$, after which λ may be extracted through solution of an elliptic equation.

6.2 Salmon's L-1 System

Salmon (1983) derived a filtered system of equations by introducing approximations directly into the Lagrangian and then applying Hamilton's principle. Our starting point is the Lagrangian function (42) in Lagrange variables (with the centrifugal term absorbed in the gravitational potential:

$$\mathscr{L} = \iint \tfrac{1}{2}\left\{\frac{\partial\mathbf{x}}{\partial\tau}\cdot\frac{\partial\mathbf{x}}{\partial\tau} + 2\mathbf{W}\cdot\frac{\partial\mathbf{x}}{\partial\tau} - gh\right\}\,da db\,.$$

As an initial simplification, we omit the first term entirely and consider the approximate Lagrangian

$$\mathscr{L}_0 = \iint \left\{\mathbf{W}\cdot\frac{\partial\mathbf{x}}{\partial\tau} - \tfrac{1}{2}g\frac{\partial(a,b)}{\partial(x,y)}\right\}\,da db\,.$$

Variations in the particle locations yield the geostrophic relationships for wind in terms of geopotential height. Since mass conservation is unaffected, the following set of equations obtains:

$$fv = g\frac{\partial h}{\partial x}\,, \quad fu = -g\frac{\partial h}{\partial y}$$

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0\,.$$

This level of approximation is too drastic for most purposes. We refine it by substituting the geostrophic expression selectively in the Lagrangian, to obtain

$$\mathscr{L}_1 = \iint \tfrac{1}{2}\left\{\mathbf{u}_\mathrm{G}\cdot\frac{\partial\mathbf{x}}{\partial\tau} + 2\mathbf{W}\cdot\frac{\partial\mathbf{x}}{\partial\tau} - gh\right\} \,da db\,. \tag{56}$$

Variations of the particle locations now yield the equation

$$\frac{d\mathbf{u}_\mathrm{G}}{dt} + \mathbf{u}_\mathrm{G}\cdot\nabla\mathbf{u}_\mathrm{A} + f\mathbf{k}\times\mathbf{u} + g\nabla h = -\frac{g}{h}\nabla B \tag{57}$$

where the quantity B is defined by

$$B = h^2\mathbf{k}\cdot\nabla\times(\mathbf{u}_\mathrm{A}/f) + \tfrac{1}{2}h^2[\mathbf{u}_\mathrm{A}\times\nabla(1/f)]\cdot\mathbf{k}\,,$$

where $\mathbf{u}_\mathrm{A} = \mathbf{u} - \mathbf{u}_\mathrm{G}$ is the ageostrophic wind. This equation contains the exact Coriolis and pressure gradient terms, but the advection $d\mathbf{u}/dt$ is approximated by neglecting the local rate of change of the ageostrophic wind, $\partial\mathbf{u}_\mathrm{A}/\partial t$, and the term $\mathbf{u}_\mathrm{A}\cdot\nabla\mathbf{u}_\mathrm{A}$. The former omission has the effect of filtering out gravity wave solutions from the system. The continuity equation is unchanged in form.

The conservation laws for this 'L-1 system' follow directly from those of the unapproximated system. The conserved energy quantity is

$$\iint \tfrac{1}{2}\left\{u_\mathrm{G}^2 + v_\mathrm{G}^2 + gh\right\} da db$$

and the geostrophic potential vorticity

$$\left[\frac{1}{h}\left(\frac{\partial v_\mathrm{G}}{\partial x} - \frac{\partial u_\mathrm{G}}{\partial y} + f\right)\right]$$

is constant for each fluid particle.

6.3 Hamilton's Principle in Euler Variables

The variational derivation of Eulerian equations is more difficult. We follow the method of Holm (1996). The action is minimized under variations of particle labels $\mathbf{a} = (a, b)$ at constant Eulerian position $\mathbf{x}$ and time t. The Lagrangian is given by

$$\mathscr{L}_\mathrm{E} = \iint \left\{\tfrac{1}{2}h\left(\mathbf{u}\cdot\mathbf{u} + 2\mathbf{W}\cdot\mathbf{u} - gh\right)\right\} \,dx dy\,. \tag{58}$$

Hamilton's principle requires that the variation of the action $\delta\int\mathscr{L}_\mathrm{E}dt$ vanish. In terms of variations of the Euler variables, this is

$$\int dt \iint dx dy \left\{h(\mathbf{u}+\mathbf{W})\cdot\delta\mathbf{u} + \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{W} - gh)\delta h\right\}\,.$$

To proceed, we require expressions for $\delta\mathbf{u}$ and δh in terms of the label variations δa and δb. Since the labels $\mathbf{a} = (a, b) = a^\alpha$ are constant following the motion, we have

$$\frac{da^\alpha}{dt} = \frac{\partial a^\alpha}{\partial t} + u\frac{\partial a^\alpha}{\partial x} + v\frac{\partial a^\alpha}{\partial y} = \partial_t a^\alpha + u^j\partial_j a^\alpha = 0\,,$$

where $\partial_t = \partial/\partial t$ and $(\partial_1, \partial_2) = (\partial/\partial x, \partial/\partial y)$. Thus, the Eulerian velocities and labels are related by

$$\partial_t a^\alpha = -(\mathcal{J})^\alpha_i u^i \qquad \text{and} \qquad u^i = -(\mathcal{J}^{-1})^i_\alpha \partial_t a^\alpha$$

where the matrix $\mathcal{J}$ and its inverse are given by

$$\mathcal{J} = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \qquad \text{and} \qquad \mathcal{J}^{-1} = \frac{1}{h}\begin{pmatrix} b_y & -a_y \\ -b_x & a_x \end{pmatrix}\,.$$

Note that the determinant of $\mathcal{J}$ is h. It then follows that the variations of $\mathbf{u}$ and h may be expressed in terms of label variations:

$$\delta u^i = -(\mathcal{J}^{-1})^i_\alpha(\partial_t\delta a^\alpha + u_j\partial_j\delta a^\alpha) \quad \text{and} \quad \delta h = h(\mathcal{J}^{-1})^i_\alpha\partial_i\delta a^\alpha\,. \tag{59}$$

By means of these relationships, it is possible to express the variation of action purely in terms of variations of the particle labels. However, considerable algebraic manipulation is required. The following relationships are easily proved:

$$\frac{d}{dt}(\mathcal{J}^{-1})^i_\alpha = (\mathcal{J}^{-1})^j_\alpha(\partial_j u^i) \qquad \text{and} \qquad \partial_i(h(\mathcal{J}^{-1})^i_\alpha) = 0\,. \tag{60}$$

By means of these, using the appropriate boundary conditions after partial integrations, and employing the continuity equation, the integrand of the action may be written eventually as

$$\left\{h(\mathcal{J}^{-1})^i_\alpha\left[\frac{du^i}{dt} + u^j(\partial_j W^i - \partial_i W^j) + g\partial_i h\right]\right\}\delta a^\alpha\,.$$

Since the label variations are arbitrary, the quantity in square brackets has to vanish at all points and times. We note that

$$u^j(\partial_j W^i - \partial_i W^j) = f(\mathbf{k}\times\mathbf{u})_i$$

so that, finally, the usual form of the momentum equation is obtained:

$$\frac{d\mathbf{u}}{dt} + f(\mathbf{k}\times\mathbf{u}) + g\nabla h = 0\,. \tag{61}$$

This is now seen as the consequence of Hamilton's principle for the shallow water system in Eulerian variables.

6.4 Balanced Eulerian Systems

We will describe a simple balanced system derived by Allen and Holm (1996) and discuss various ways in which it can be generalized. The Lagrangian (58) may be written

$$\mathscr{L}_{\mathrm{E}} = \iint \left\{ h(\mathbf{W}+\mathbf{u})\cdot\mathbf{u} - \tfrac{1}{2}gh^2 - \tfrac{1}{2}h\mathbf{u}\cdot\mathbf{u} \right\} \, dxdy\,. \tag{62}$$

We assume the Rossby number ϵ is small and partition the wind field into geostrophic and ageostrophic components: $\mathbf{u} = \mathbf{u}_{\mathrm{G}} + \epsilon\mathbf{u}_{\mathrm{A}}$. Keeping terms up to $O(\epsilon)$, the Lagrangian may be written

$$\mathscr{L}_{\mathrm{E}} = \iint \left\{ h(\mathbf{W}+\mathbf{u}_{\mathrm{G}})\cdot\mathbf{u} - \tfrac{1}{2}gh^2 - \tfrac{1}{2}h(\mathbf{u}_{\mathrm{G}}\cdot\mathbf{u}_{\mathrm{G}}) \right\} \, dxdy\,. \tag{63}$$

Then, following arguments similar to those in the preceding section, Hamilton's principle (with variations of Lagrangian labels at constant Eulerian positions and time) results in the following system:

$$\begin{aligned} \frac{\partial \mathbf{u}_{\mathrm{G}}}{\partial t} + [\nabla\times(\mathbf{W}+\mathbf{u}_{\mathrm{G}})]\times\mathbf{u} + \nabla B = 0 \\ \frac{\partial h}{\partial t} + \nabla\cdot h\mathbf{u} = 0 \end{aligned}$$

where the Bernoulli function B is defined as

$$B = \left[gh + \tfrac{1}{2}\mathbf{u}_{\mathrm{G}}\cdot\mathbf{u}_{\mathrm{G}} + (1/\mathscr{F})\mathbf{k}\cdot\nabla\times[h(\mathbf{u}-\mathbf{u}_{\mathrm{G}})] \right] ,$$

where $\mathscr{F}$ is the Froude number. This system is closely related to Salmon's L-1 system, derived using Hamilton's principle in Lagrange variables (with variations in particle paths at fixed time and Lagrangian labels).

A similar system may be derived much more simply by replacing $\mathbf{u}$ by $\mathbf{u}_{\mathrm{G}}$ as the variable operated on by d/dt in the momentum equation (61). This was the approach adopted by Eliassen (1948) to derive the system known as the geostrophic momentum approximation. This system is equivalent to order $O(\epsilon)$ with the one obtained here. However, such a cavalier approach has its pitfalls: the system derived through Hamilton's principle conserves total energy and potential vorticity of fluid parcels, whereas there is no guarantee that the system derived by *ad-hoc* methods respects these principles. The additional terms of order $O(\epsilon^2)$ which are retained in ths Hamiltonian approach are precisely those needed for energy and potential vorticity conservation.

Allen and Holm (1996) show that more accurate systems may be derived by retaining terms of higher order in the Rossby number. Holm (1996) presents a baroclinic system which he calls the Hamiltonian balance equations (HBE). Finally, Holm, Marsden and Ratiu (2002) present a hierarchy of equation systems based on successively stronger approximations to the Lagrangian. Starting with the Euler equations for three-dimensional flow, incompressibility is enforced through a constraint in which the Lagrange multiplier turns out to be the pressure. When the buoyancy is constrained to be small, the Euler-Boussinesq system is obtained. The hydrostatic primitive equations follow upon assuming that the aspect ratio is small. Expansion in the Rossby number then leads to a sequence of balanced systems, HBE, Salmon's L-1 system and the classical quasi-geostrophic equations. At all stages, the Hamiltonian formalism ensures that appropriate energy and potential vorticity principles apply, This is so because the approximations do not disrupt the symmetry properties of the Hamiltonian.

7 Summary

The powerful methods of Hamiltonian mechanics have significant benefits when applied to geophysical fluid dynamics problems. The methods have been introduced first in the context of finite-dimensional mechanical systems, with illustration of several key ideas through application to the elastic pendulum or swinging spring. The Lagrangian formulation of the fluid dynamical equations has been presented, and compared to the more familiar Eulerian formulation. The process of reduction from Lagrangian to Eulerian formulation has been described. The rotating shallow water equations have been taken as a simple example of a geophysical equation system, and various Hamiltonian forms of the equations have been presented. Finally, the passage from 'exact' to 'approximate' equations, through application of constraints with Lagrange multipliers, or by other means, has been described, and a number of balanced systems resulting from the process have been presented.

The methods developed in this brief introduction are of very general applicability. There is a wide range of possibilities for deriving approximate systems of equations with varying levels of accuracy. One particular sequence of filtered equations was proposed long ago by Hinkelmann (1969). He proposed a general method for defining initial data for the primitive equations. He argued that the observed mass and wind fields should be adjusted so that

$$\frac{d^{\,n}(\nabla\cdot\mathbf{V})}{dt^{\,n}} = 0 \qquad \text{and} \qquad \frac{d^{\,n+1}(\nabla\cdot\mathbf{V})}{dt^{\,n+1}} = 0. \tag{2.2}$$

That is, these two conditions should be used to derive diagnostic relationships which the initial data are then required to satisfy. As an alternative, he pointed out that the two conditions could be used to replace two prognostic equations by diagnostic relationships, yielding a general filtered system. The case $n = 0$ yields the quasi-geostrophic equations, while the case $n = 1$ leads to the slow equations (Lynch, 1989). It is proposed, in a future investigation, to implement the Hinkelmann hierarchy using the Hamiltonian formalism.

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References

Allen, J. S. and Darryl D. Holm, 1996: Extended-geostrophic Hamiltonian models for rotating shallow water motion. *Physica D*, **98**, 229–249.

Bokhove, O., 2000: On hydrostatic flows in isentropic coordinates. *J. Fluid Mech.*, **402**, 291–310.

Bokhove, O., 2002: Balanced models in geophysical fluid dynamics: Hamiltonian formulation, constraints and formal stability. Pp. 1–63 in *Large-Scale Atmosphere-Ocean Dynamics: Vol II: Geometric Methods and Models.* Ed. J. Norbury and I. Roulstone. Cambridge University Press.

Bretherton, F. P., 1971: The general linearised theory of wave propagation. *Lectures in Appl. Maths.*, **13**, 61–102.

Casimir, H. B. G., 1931: *Rotation of a rigid body in quantum Mechanics.* Thesis, J.B.Wolters Uitgevers-Maatschappij, N.V.Groningen, den Haag, Batavia.

Eliassen, A, 1948: The quasi-static equations of motion with pressure as independent variable. *Geofys. Publ.*, **17**, No. 3, 44pp.

Goldstein, H., C. Poole and J. Safko, 2002: *Classical Mechanics*. 3rd Edn., Addison-Wesley. 638pp.

Hinkelmann, K. H., Primitive Equations. Pp. 306–375 in *Lectures in numerical short range weather prediction.* Regional Training Seminar, Moscow. World Meteorological Organization, Geneva, WMO No. 297.

Holm, Darryl D., 1996: Hamiltonian balance equations. *Physica D*, **98**, 379–414.

Holm, Darryl D. and Peter Lynch, 2002: Stepwise Precession of the Resonant Swinging Spring. Accepted for publication in *SIAM Journal on Applied Dynamical Systems*

Holm, Darryl D., J. E. Marsden and Tudor S. Ratiu, 2002: The Euler-Poincaré Equations in Geophysical Fluid Dynamics. To appear in *Large-Scale Atmosphere-Ocean Dynamics: Vol I.* Ed. J. Norbury and I. Roulstone. Cambridge University Press.

José, J. V. and E. J. Saletan, 1998: *Classical Dynamics: a Contemporary Approach*. Cambridge Univ. Press, 670pp.

Lynch, Peter, 1989: The slow equations. *Q. J. Roy. Meteor. Soc.* **115**, 201–219.

Lynch, Peter, 2002a: Resonant motions of the three-dimensional elastic pendulum. *Intl. J. Nonlin. Mech.*, **37**, 345–367.

Lynch, Peter, 2002b: The Swinging Spring: a Simple Model for Atmospheric Balance. Pp. 64-108 in *Large-Scale Atmosphere-Ocean Dynamics: Vol II: Geometric Methods and Models.* Ed. J. Norbury and I. Roulstone. Cambridge University Press.

Lynch, Peter, 2002c: Resonant Rossby Wave Triads and the Swinging Spring. Submitted to *Bull. Amer. Met. Soc.*

Marsden, Jerrold E. and Tudor S. Ratiu, 1999: *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems.* Springer Verlag, New York, Second edition, 582 pp.

Morrison, P. J., 1982: Poisson brackets for fluids and plasmas. Pp. 13–46 in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*. Ed. M. Tabor and Y.M.Treve. AIP Conf. Proc., No. 88, Am. Inst. Phys., New York.

Morrison, P. J., 1998: Hamiltonian description of the ideal fluid. *Rev. Mod. Physics*, **70**, 467–521.

Morrison, P. J. and J. M. Greene, 1980: Non-canonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Rev. Lett.*, **45**, 790–794.

Roulstone, I. and S. J. Brice, 1995: On the Hamiltonian formulation of the quasi-hydrostatic equations. *Q. J. Roy. Met. Soc.*, **121**, 927–936.

Salmon, Rick, 1983: Practical use of Hamilton's principle. *J. Fluid Mech.*, **132**, 431–444.

Salmon, Rick, 1988a: Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.*, **20**, 225–256.

Salmon, Rick, 1988b: Semigeostrophic theory as a Dirac-bracket projection. *J. Fluid Mech.*, **196**, 345–358.

Salmon, Rick, 1998: *Lectures on Geophysical Fluid Dynamics.* Oxford Univ. Press, 378pp.

Shepherd, T. G., 1990: Symmetries, conservation laws and Hamiltonian structure in geophysical fluid dynamics. *Adv. Geophys.*, **32**, 287–338.

Sudarshan, E. C. G. and N. Mukunda, 1974: *Classical Dynamics: a Modern Perspective.* John Wiley & Sons, 615pp.

Swaters, G. E., 2000: *Introduction to Hamiltonian Fluid Dynamics and Stability Theory*. Chapman & Hall/CRC, 274pp.

Weiss, J., 1991: The dynamics of enstrophy transfer in two-dimensional hydrodynamics. *Physica D*, **48**, 273–294.