Parity and Partition of the Rational Numbers

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The natural numbers $\mathbb{N}$ split nicely into two subsets, the odd and even numbers

$$\mathbb{N}_{\mathrm{O}} = \{1, 3, 5, 7, \dots\}, \qquad \mathbb{N}_{\mathrm{E}} = \{2, 4, 6, 8, \dots\}.$$

Stopping at some number $2N$, the odd and even numbers are equinumerous. Stopping at $2N + 1$, the odds are slightly ahead, but as N gets larger, the ratio of odd to even numbers tends to 1. So, we can say informally that there are the same number of odd and even integers. This will be made precise below by defining densities for the sets $\mathbb{N}_{\mathrm{O}}$ and $\mathbb{N}_{\mathrm{E}}$. Similar arguments apply to the integers $\mathbb{Z}$, which split into two subsets

$$\mathbb{Z}_{\mathrm{O}} = \{\dots, -3, -1, +1, +3, +5, \dots\}$$
$$\mathbb{Z}_{\mathrm{E}} = \{\dots, -4, -2, \ \ 0, +2, +4, \dots\}.$$

The integers form an abelian group $(\mathbb{Z}, +)$ under addition. The even numbers form an additive subgroup of $(\mathbb{Z}, +)$, with index $[\mathbb{Z} : \mathbb{Z}_{\mathrm{E}}] = 2$ and two cosets $\mathbb{Z}_{\mathrm{E}}$ and $\mathbb{Z}_{\mathrm{E}} + 1 = \mathbb{Z}_{\mathrm{O}}$. This definition provides a bijection between the two cosets, which have the same cardinality.

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Table 1. Addition table (left) and multiplication table (right) for $\mathbb{Z}$.

+	**even**	**odd**
even	*even*	*odd*
odd	*odd*	*even*

×	**even**	**odd**
even	*even*	*even*
odd	*even*	*odd*

Parity

The distinction between odd and even numbers is called *parity*. The even/odd concept is defined only for the integers. The distinction does not apply to fractions or irrational numbers, but one may wonder if there is a natural way to extend the concept of parity to larger sets of numbers.

What characteristics might one require of such an extension? The definition would have to agree with the traditional definition for the integers, so 5 would continue to be odd and 10 even. In addition, the usual "rules of parity" might be required:

1. The sum of two even numbers is even; the product is even.
2. The sum of two odd numbers is even; the product is odd.
3. The sum of an even and an odd number is odd; the product is even.
4. An odd number plus 1 is even; an even number plus 1 is odd.

Table 1 shows the effects of addition and multiplication on the ring of integers.

If the concept of parity is extended to larger sets of numbers, some of the properties indicated above may have to be sacrificed. We might define a rational number $q = m/n$ (in reduced form) to be even if the numerator m is even and odd if m is odd. But then $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, meaning that two odd rationals would add to yield another odd one.

It is easy to distinguish between even rationals and those that are not even:

$$\text{For any rational number } q = \frac{m}{n}, \qquad \begin{cases} q \text{ is } \textit{even} \text{ if } m \text{ is even,} \\ q \text{ is } \textit{uneven} \text{ if } m \text{ is odd,} \end{cases}$$

where m and n are relatively prime integers, $(m, n) = 1$.[1] However, the two classes, "even" and "uneven," do not respect the rules of parity, so something better is required.

A three-way split

There is a simple way of separating the rational numbers into three subsets:

$$\text{For any rational } q = \frac{m}{n}, \qquad \begin{cases} q \text{ has parity } \textit{even} \text{ if } m \text{ is even and } n \text{ is odd,} \\ q \text{ has parity } \textit{odd} \text{ if } m \text{ is odd and } n \text{ is odd,} \\ q \text{ has parity } \textit{none} \text{ if } m \text{ is odd and } n \text{ is even.} \end{cases}$$

The term *none* is an initialism for "neither odd nor even." Corresponding to this three-way partition, we define three subsets of the rationals:

$$\text{Even: } \mathbb{Q}_{\mathrm{E}} = \{q \in \mathbb{Q} : q = \tfrac{2m}{2n+1} \text{ for some } m, n \in \mathbb{Z}\}$$

[1]From here onwards, all fractions $\frac{a}{b}$ will be considered to be in reduced form, with $(a, b) = 1$.

$$\text{Odd: } \mathbb{Q}_\text{O} = \{q \in \mathbb{Q} : q = \tfrac{2m+1}{2n+1} \text{ for some } m, n \in \mathbb{Z}\}$$

$$\text{None: } \mathbb{Q}_\text{N} = \{q \in \mathbb{Q} : q = \tfrac{2m+1}{2n} \text{ for some } m, n \in \mathbb{Z}\}.$$

These three mutually disjoint sets comprise the rationals: $\mathbb{Q} = \mathbb{Q}_\text{E} \uplus \mathbb{Q}_\text{O} \uplus \mathbb{Q}_\text{N}$ (where $\uplus$ denotes the disjoint union). It is immediately obvious that $\mathbb{Z}_\text{E} \subset \mathbb{Q}_\text{E}$ and $\mathbb{Z}_\text{O} \subset \mathbb{Q}_\text{O}$, confirming that the definition of parity for the rationals is an extension of the usual meaning for the integers. We see that the even and odd rationals respect the four "rules of parity" listed above.

"Twice as many uneven as even fractions". The rational numbers are countable: they can be put into one-to-one correspondence with the natural numbers. We can list all rationals in (0, 1) in a sequence where, for each n in turn, all (new) numbers m/n with $m < n$ are listed in order. For $n_\text{max} = 8$ we have

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}. \tag{1}$$

Rearrangement in increasing order of magnitude gives the Farey sequence F_8.

A MATHEMATICA program was written to count the proportion of rationals in each parity class in the interval (0, 1), with denominators less than or equal to n, for a range of cutoff values $n \leqslant n_\text{max}$. The ratios are plotted in Figure 1. As more rationals are included, the ratios of numbers with parity even, odd, and none all tend to the limit $\frac{1}{3}$. Colloquially, there are an equal number of rationals with parity even, odd, and none, and "twice as many uneven as even rationals."

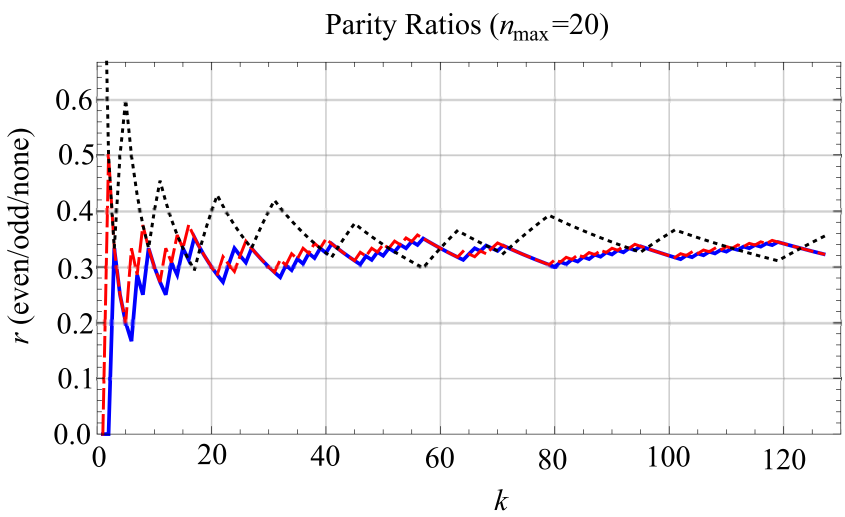

Figure 1. Parity ratio r for rationals m/n of parity even (solid line, blue online), odd (dashed line, red online) and none (dotted line, black online) for $n_\text{max} = 20$. The horizontal axis k is the count of rational numbers used to compute the ratios. For $n_\text{max} = 20$, there are 127 rationals in (0, 1).

The density of subsets of $\mathbb{N}$

In pure set-theoretic terms, the set of even natural numbers is "the same size" as the set of all natural numbers; both are infinite countable sets. However, cardinality is a blunt instrument: with the usual ordering, every second natural number is even and, intuitively, we feel that there are half as many even numbers as natural numbers. The concept of *density* provides a means of expressing the relative sizes of sets that is more discriminating than cardinality.

Density—also called natural or asymptotic density—is defined for many interesting subsets of $\mathbb{N}$, although not for all subsets. Assume a subset A of $\mathbb{N}$ is enumerated as $\{a_1, a_2, \dots\}$. We define the density of A in $\mathbb{N}$ as the limit, if it exists,

$$\rho_{\mathbb{N}}(A) = \lim_{n\to\infty} \frac{|\{a_k : a_k \leqslant n\}|}{n}.$$

Thus, if the fraction of elements of A among the first n natural numbers converges to a limit $\rho_{\mathbb{N}}(A)$ as n tends to infinity, then A has density $\rho_{\mathbb{N}}(A)$ [7, p. 270]. More generally, if $A = \{a_1, a_2, a_3, \dots\}$ is a subset of a countable set X enumerated as $\{x_1, x_2, x_3, \dots\}$, we define the density of A in X—if it exists—as

$$\rho_X(A) = \lim_{n\to\infty} \frac{|A \cap \{x_1, x_2, \dots, x_n\}|}{n}. \tag{2}$$

For $X = \mathbb{N}$, we usually write $\rho_{\mathbb{N}}(A)$ as $\rho(A)$. For $A = \mathbb{N}_{\mathrm{E}}$ or $A = \mathbb{N}_{\mathrm{O}}$, we have $\rho(A) = \frac{1}{2}$, as might be expected. This is consistent with our intuitive notion that 50% of the natural numbers are even and 50% are odd.

Let us now rearrange the natural numbers into a set F such that there are *twice as many even as odd numbers in* F. We reorder $\mathbb{N}$ so that each odd number is followed by two even ones:

$$F = \{1, 2, 4, 3, 6, 8, 5, 10, 12, \dots, 2n-1, 4n-2, 4n, \dots\}.$$

It is easy to see that $\rho_F(\mathbb{N}_{\mathrm{E}}) = \frac{2}{3}$ and $\rho_F(\mathbb{N}_{\mathrm{O}}) = \frac{1}{3}$. Proceeding further, we can construct a set H in which the n-th odd number is followed by n even numbers. We find that $\rho_H(\mathbb{N}_{\mathrm{E}}) = 1$, so that "almost all the elements of H are even."

These examples make it clear that density depends strongly on the ordering of the reference set. Our intuition is guided by the usual (natural) ordering of the natural numbers, and the alternation between odd and even numbers leads us to the conclusion that, somehow, they are equal in number, each comprising "half" of the set of natural numbers. Density relative to $\mathbb{N}$ is consistent with this intuition.

With the ordering $\{0, +1, -1, +2, -2, \dots\}$ of the integers, the densities defined by (2) are

$$\rho_{\mathbb{Z}}(\mathbb{Z}_{\mathrm{E}}) = \tfrac{1}{2}, \qquad \rho_{\mathbb{Z}}(\mathbb{Z}_{\mathrm{O}}) = \tfrac{1}{2}.$$

We will prove that, for the rational numbers with the Calkin-Wilf and Stern-Brocot orderings illustrated in Figure 3,

$$\rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{E}}) = \rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{O}}) = \rho_{\mathbb{Q}}(\mathbb{Q}_{\mathrm{N}}) = \tfrac{1}{3}.$$

Partitioning the rationals

In Table 2 we show the results of adding and multiplying numbers from the three parity classes. The most important thing to notice is that, if we confine attention to only the even and odd rationals, the tables are identical to the addition and multiplication tables for $\mathbb{Z}$ (Table 1). The entry "any" in the tables indicates a sum or product that is a ratio of two even numbers and that may, after reduction, be in any of the three parity classes. Examining the left panel of Table 2, we see that $(\mathbb{Q}_{\mathrm{E}}, +)$ is an additive

Table 2. Addition table (left) and multiplication table (right) for $\mathbb{Q}$. The entry "*any*" indicates that the result may be in any of the three parity classes.

+	even	odd	none
even	*even*	*odd*	*none*
odd	*odd*	*even*	*none*
none	*none*	*none*	*any*

×	even	odd	none
even	*even*	*even*	*any*
odd	*even*	*odd*	*none*
none	*any*	*none*	*none*

(normal) subgroup of $(\mathbb{Q}, +)$. In Table 2 (right panel) we show the results of multiplying numbers from the three parity classes. Restricting attention to the even and odd rationals only—omitting those with no parity—we define

$$\mathbb{Q}_P := \mathbb{Q}_E \uplus \mathbb{Q}_O.$$

This is the set of all rationals whose denominators are odd numbers in $\mathbb{Z}$. It is closed under addition and multiplication and forms a subring of the field $\mathbb{Q}$. Moreover, since there are no divisors of zero, $\mathbb{Q}_P$ is an integral domain [4, p. 228]. Although $\mathbb{Q}_P$ is not an ideal of $\mathbb{Q}$ (fields do not have nontrivial proper ideals), it is a (normal) subgroup of $(\mathbb{Q}, +)$. So, we may enquire about its index $[\mathbb{Q} : \mathbb{Q}_P]$ and its quotient group $\mathbb{Q}/\mathbb{Q}_P$.

Somewhat out of context, we mention the easily-proved observation that all three parity classes, $\mathbb{Q}_E$, $\mathbb{Q}_O$, and $\mathbb{Q}_N$, are (topologically) dense in the rationals.

2-Adic valuation and the "degree of evenness".

All multiples of 2 are even, but some are more even than others.

The p-adic valuation—or p-adic order [6, p. 20]—of an integer n, defined for all prime numbers p, is the exponent of the largest power of p that divides n:

$$\nu_p(n) := \begin{cases} \max\{k \in \mathbb{N} \cup \{0\} : p^k \mid n\} & \text{for } n \neq 0 \\ \infty & \text{for } n = 0. \end{cases}$$

It may be extended to the rational numbers m/n:

$$\nu_p\left(\frac{m}{n}\right) = \nu_p(m) - \nu_p(n).$$

It is easily proved that, for any rationals q_1 and q_2,

$$\nu_p(q_1 + q_2) \geqslant \min\{\nu_p(q_1), \nu_p(q_2)\}, \tag{3}$$

with equality holding if $\nu_p(q_1) \neq \nu_p(q_2)$.

We shall be concerned exclusively with the case $p = 2$. We note that $\mathbb{Q}_P = \{q \in \mathbb{Q} : \nu_2(q) \geqslant 0\}$ and $\mathbb{Q}_E = \{q \in \mathbb{Q} : \nu_2(q) > 0\}$. The "degree of evenness" of a number can be expressed in terms of the 2-adic valuation. For even integers, $\nu_2(n) > 0$; for odd integers, $\nu_2(n) = 0$. By convention, $\nu_2(0) = \infty$ (since zero is divisible by every power of 2).

If we write a rational number q in the form $2^k(2m + 1)/(2n + 1)$ with $k \in \mathbb{Z}$, then $\nu_2(q) = k$. Odd rationals have order 0 and rationals with no parity have negative 2-adic order. In particular, half an odd integer has 2-adic order equal to -1. In summary,

For q rational with parity *even*, $\nu_2(q) > 0$,

$$\text{For } q \text{ rational with parity } \textit{odd}, \; \nu_2(q) = 0,$$
$$\text{For } q \text{ rational with parity } \textit{none}, \; \nu_2(q) < 0.$$

The 2-adic order clearly identifies the parity classes of the rationals, and it provides a means of partitioning them into finer-grain parity classes. The resulting partition reveals a wealth of algebraic structure. For all $k \in \mathbb{Z}$, we define the set of all rational numbers with 2-adic valuation k:

$$Q_k = \{q \in \mathbb{Q} : \nu_2(q) = k\} \qquad \text{and} \qquad Q_\infty = \{0\}.$$

The union of all the Q-sets comprises the entire set of rationals

$$\mathbb{Q} = \{0\} \uplus \biguplus_{k=-\infty}^{\infty} Q_k.$$

We illustrate the subsets Q_k in Figure 2. The vertical axis is the 2-adic valuation ν_2. Each subset Q_k is represented by a horizontal dotted line. We remark that all odd rationals are in Q_0, and all even rationals are in $\bigcup_{k>0} Q_k$. For all $K \in \mathbb{Z}$, we define

$$\mathbb{Q}_\mathrm{K} := \{0\} \uplus \biguplus_{k \geqslant K} Q_k \tag{4}$$

and observe that $\mathbb{Q}_\mathrm{K}$ is an additive subgroup of $\mathbb{Q}$. We write this as $\mathbb{Q}_\mathrm{K} \trianglelefteq \mathbb{Q}$. In particular, $\mathbb{Q}_0 = \mathbb{Q}_\mathrm{P}$ and $\mathbb{Q}_1 = \mathbb{Q}_\mathrm{E}$. There is an infinite chain of subgroups, starting with $\mathbb{Q}_\infty = \{0\}$ and extending through all the $\mathbb{Q}_K$ groups to the full group of rationals:

$$\mathbb{Q}_\infty = \{0\} \trianglelefteq \cdots \trianglelefteq \mathbb{Q}_2 \trianglelefteq \mathbb{Q}_1 \trianglelefteq \mathbb{Q}_0 \trianglelefteq \mathbb{Q}_{-1} \trianglelefteq \mathbb{Q}_{-2} \trianglelefteq \cdots \trianglelefteq \mathbb{Q}.$$

Dyadic rational numbers. A dyadic rational is a number that can be expressed as a fraction whose denominator is a power of two. The usual definition of the dyadic rational numbers [1, p. 122] is

$$\mathbb{D} = \left\{ \frac{z}{2^m} : z \in \mathbb{Z}, m \in \mathbb{Z} \right\}.$$

Note that the integers are included in the set of dyadic rationals. A convenient alternative definition is

$$\mathbb{D} = \{2^k(2\ell - 1) : k \in \mathbb{Z}, \ell \in \mathbb{Z}\} \uplus \{0\},$$

since all the numbers of the form $2^k(2\ell - 1)$ are in Q_k. Moreover, the expression of each number in this form is unique. We also define the sets

$$D_k = \{2^k(2\ell - 1) : \ell \in \mathbb{Z}\} \qquad \text{and} \qquad D_\infty = \{0\},$$

and note that $\mathbb{D} = \biguplus_k D_k \uplus D_\infty$. We see that $D_0 = \mathbb{Z}_\mathrm{O}$ and $\biguplus_{k>0} D_k = \mathbb{Z}_\mathrm{E}$.

The dyadics correspond to all real numbers with finite binary expansions and also to the set of surreal numbers born on finite days [3, p. 29]. The dyadic rational numbers form a ring between the ring of integers and the field of rational numbers:

$$\mathbb{Z} \trianglelefteq \mathbb{D} \trianglelefteq \mathbb{Q}.$$

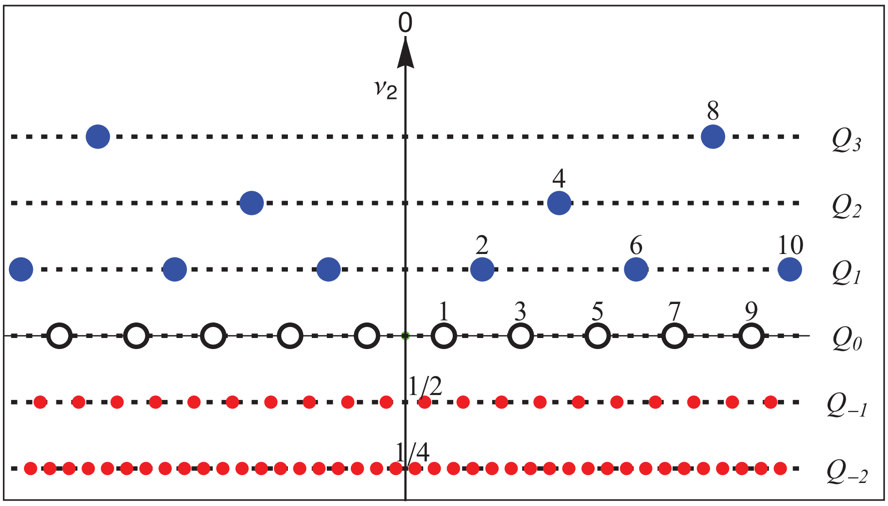

Figure 2. Partition of the rational numbers. The vertical axis is the 2-adic valuation ν_2. Each (dense) subset Q_k is represented by a horizontal dotted line. The sets D_k are indicated by the marked points in Q_k. The totality of these comprises the dyadic rationals $\mathbb{D}$.

The sets D_k are indicated in Figure 2 by the marked points in Q_k. For each k, $D_k \subset Q_k$. The circles at level $k = 0$ are the odd integers. The large dots (blue online) at positive k-levels are the even integers. The small dots (red online) at each negative level $k < 0$ are the dyadic fractions, with odd numerator and denominator 2^k. Zero sits, like an angel, on top of the tree.

By analogy with the definition (4) of the $\mathbb{Q}_K$-sets, we construct a countable infinity of subgroups of $\mathbb{D}$:

$$\mathbb{D}_K := \{0\} \uplus \biguplus_{k \geqslant K} D_k .$$

Particular cases of the $\mathbb{D}$-sets include

$$\mathbb{D}_{-\infty} = \mathbb{D}, \qquad \mathbb{D}_{-1} = \tfrac{1}{2}\mathbb{Z}, \qquad \mathbb{D}_0 = \mathbb{Z}, \qquad \mathbb{D}_1 = \mathbb{Z}_{\mathrm{E}}, \qquad \mathbb{D}_\infty = \{0\}.$$

There is an infinite chain of subgroups starting with $\mathbb{D}_\infty$ and extending through all the $\mathbb{D}_K$ groups to the full group of dyadic rationals:

$$\mathbb{D}_\infty = \{0\} \trianglelefteq \ \cdots \trianglelefteq \ \mathbb{D}_2 \trianglelefteq \mathbb{D}_1 \trianglelefteq \mathbb{D}_0 \trianglelefteq \mathbb{D}_{-1} \trianglelefteq \mathbb{D}_{-2} \trianglelefteq \ \cdots \trianglelefteq \ \mathbb{D}.$$

Readers familiar with the theory of p-adic numbers may wish to show that $\mathbb{Q}_{\mathrm{P}}$ is the ring of rational-valued 2-adic integers, $\mathbb{Q} \cap \mathbb{Z}_2$, and the dyadic rational numbers may be expressed as

$$\mathbb{D} = \mathbb{Q} \cap \bigcap_{p \text{ odd}} \mathbb{Z}_p .$$

Cosets of $\mathbb{Q}_{\mathrm{P}}$ in $\mathbb{Q}$

In the following section, we show that any two rationals q_1 and q_2 with distinct, negative 2-adic orders are representatives of distinct cosets: $q_1 + \mathbb{Q}_{\mathrm{P}} \neq q_2 + \mathbb{Q}_{\mathrm{P}}$. Thus, if $q_1 + \mathbb{Q}_{\mathrm{P}} = q_2 + \mathbb{Q}_{\mathrm{P}}$, then $\nu_2(q_1) = \nu_2(q_2)$. Consequently, there is at least one coset for each $k < 0$ and therefore an infinite number of cosets. However, it is clear that $\nu_2(q_1) = \nu_2(q_2)$ does not imply equality of cosets; consider, for example, $q_1 = \frac{1}{4}$ and $q_2 = \frac{3}{4}$, since $(q_2 - q_1) = \frac{1}{2} \notin \mathbb{Q}_{\mathrm{P}}$.

We now investigate the cosets $q + \mathbb{Q}_{\mathrm{P}}$ in $\mathbb{Q}/\mathbb{Q}_{\mathrm{P}}$. First, we note that if $q \in Q_{-k}$ then $q + \mathbb{Q}_{\mathrm{P}} \subset Q_{-k}$, but there is no guarantee that $q + \mathbb{Q}_{\mathrm{P}}$ is equal to Q_{-k}. Suppose q_1 and q_2 are in Q_{-k} for some $k > 0$. If they represent the same coset then $q_1 - q_2 \in \mathbb{Q}_{\mathrm{P}}$. However, it is easily seen that $\nu_2(q_1 - q_2)$ may assume any value greater than $-k$:

$$q_1 - q_2 = 2^{\ell - k}\frac{o_1}{o_2} \quad \text{for} \quad \ell > 0$$

(where o_1 and o_2 are odd integers). Thus, $q_1 - q_2$ may be in any of the following sets:

$$Q_{-k+1}, Q_{-k+2}, Q_{-k+3}, \ldots, Q_{-1}, \mathbb{Q}_{\mathrm{P}}.$$

Clearly, $q_1 - q_2 \in \mathbb{Q}_{\mathrm{P}}$ if and only if $\ell \geqslant k$, whence

$$q_1 + \mathbb{Q}_{\mathrm{P}} = q_2 + \mathbb{Q}_{\mathrm{P}} \qquad \text{iff} \qquad \ell \geqslant k.$$

For each $k > 0$, we define a set of values

$$q_{k,\ell} = 2^{-k}(2\ell - 1) \in Q_{-k} \quad \text{for} \quad \ell = 1, 2, 3, \ldots, 2^{k-1}.$$

We note that these are the first 2^{k-1} positive values in D_{-k}. We show in the following section that these are representatives of 2^{k-1} cosets, which are all distinct and which provide a disjoint partition of Q_{-k}. This analysis provides explicit expressions for each of the infinite set of cosets of $\mathbb{Q}_{\mathrm{P}}$ in $\mathbb{Q}$:

$$q_{k,\ell} + \mathbb{Q}_{\mathrm{P}}, \qquad \ell = 1, 2, 3, \ldots, 2^{k-1}, \quad k = 1, 2, \ldots.$$

The diagram in Figure 2 has a scaling invariance: if the horizontal axis is stretched by a factor of 2 and the diagram translated one unit in the vertical, the dyadic rationals occupy the same set of points. We have chosen to analyze the quotient group $\mathbb{Q}/\mathbb{Q}_{\mathrm{P}}$. However, a similar analysis could be done for any subgroup $\mathbb{Q}_{\mathrm{K}}$, with directly analogous results.

Density of Q_k: a heuristic discussion. The set $Q_{-1} = \frac{1}{2} + \mathbb{Q}_{\mathrm{P}}$ is a coset of $\mathbb{Q}_{\mathrm{P}}$. It can be visualized as a copy of $\mathbb{Q}_{\mathrm{P}}$ shifted by a distance $\frac{1}{2}$. We argue heuristically that Q_{-1} is "as dense as $\mathbb{Q}_{\mathrm{P}}$".

More generally, for any k, there is a natural correspondence between elements of Q_k and elements of Q_{k-1}:

$$\frac{1}{2^k}\left(\frac{2m+1}{2n+1}\right) \in Q_k \longleftrightarrow \frac{1}{2^{k-1}}\left(\frac{2m+1}{2n+1}\right) \in Q_{k-1}.$$

Thus, Q_{k-1} may be visualized as a compressed version of Q_k. Since Q_{k-1} is "twice as dense as Q_k", we may argue that we should have twice as many cosets in Q_{k-1} as there are in Q_k. This is consistent with what is proved rigorously below.

Formal proof of the coset structure for $\mathbb{Q}_\mathrm{P}$

In this section, we give rigorous proofs of some of the results considered heuristically in the discussion above. Lemmas 1 and 2 give conditions for cosets to be equal. Proposition 1 gives explicit representatives $q_{k,\ell}$ for each of the distinct cosets. In the following, we abbreviate the 2-adic valuation ν_2 to ν.

Lemma 1. *Suppose $q_1, q_2 \in \mathbb{Q}_\mathrm{N}$ and $q_1 + \mathbb{Q}_\mathrm{P} = q_2 + \mathbb{Q}_\mathrm{P}$. Then $\nu(q_1) = \nu(q_2)$.*

Proof. For the cosets to be equal, we must have $q_1 - q_2 \in \mathbb{Q}_\mathrm{P}$. Suppose that $\nu(q_1) = -k$ and $\nu(q_2) = -\ell$ with $k \neq \ell \in \mathbb{N}$. Without loss of generality, we may assume that $\ell = k + d$ with $d > 0$. Then, using (3),

$$\nu(q_1 - q_2) = \min(\nu(q_1), \nu(-q_2)) = \min(\nu(q_1), \nu(q_2)) = \min(-\ell, -k) = -\ell.$$

Thus, $q_1 - q_2 \in Q_{-\ell}$, so that $q_1 + \mathbb{Q}_\mathrm{P} \neq q_2 + \mathbb{Q}_\mathrm{P}$. Consequently, a necessary condition for equality of the cosets is that q_1 and q_2 have the same 2-adic valuation, $\nu(q_1) = \nu(q_2)$. ■

The next lemma strengthens this to a necessary and sufficient condition.

Lemma 2. *Let q_1 and q_2 be in $\mathbb{Q}_\mathrm{N}$. Then $q_1 + \mathbb{Q}_\mathrm{P} = q_2 + \mathbb{Q}_\mathrm{P}$ if, and only if, $\nu(q_1) = \nu(q_2) = -k$ for some $k \in \mathbb{N}$ and $\nu(a_1b_2 - a_2b_1)) \geqslant k$, where $q_1 = a_1/(2^k b_1)$ and $q_2 = a_2/(2^k b_2)$.*

Proof. Let q_1 and q_2 be in $\mathbb{Q}_\mathrm{N}$. Suppose that $q_1 + \mathbb{Q}_\mathrm{P} = q_2 + \mathbb{Q}_\mathrm{P}$. Lemma 1 tells us that $\nu(q_1) = \nu(q_2) = -k$ for some $k \in \mathbb{N}$. We may thus write

$$q_1 = \frac{a_1}{2^k b_1} \quad \text{and} \quad q_2 = \frac{a_2}{2^k b_2}, \tag{5}$$

where a_i, b_i are odd, and so

$$q_1 - q_2 = \frac{a_1 b_2 - a_2 b_1}{2^k b_1 b_2}. \tag{6}$$

By hypothesis, this is an element of $\mathbb{Q}_\mathrm{P}$, which implies $2^k | (a_1b_2 - a_2b_1)$, as asserted.

For the converse, if $\nu(q_1) = \nu(q_2) = -k$, then (5), and hence (6), holds. The condition $\nu(a_1b_2 - a_2b_1) \geqslant k$ then implies $\nu(q_1 - q_2) \geqslant 0$, and so $q_1 - q_2 \in \mathbb{Q}_\mathrm{P}$. ■

Proposition 1. *For each $k \in \mathbb{N}$, let $q_{k,\ell} = (2\ell - 1)/2^k \in Q_{-k}$ for $\ell = 1, \ldots, 2^{k-1}$. These numbers generate 2^{k-1} distinct $\mathbb{Q}_\mathrm{P}$-cosets, which comprise all the cosets of $\mathbb{Q}_\mathrm{P}$ by elements of Q_{-k}.*

Proof. It is clear that $q_{k,\ell} = (2\ell - 1)/2^k$ lies in Q_{-k}. If two of these numbers, $q_{k,\ell}$ and $q_{k,\ell'}$ say, generate the same coset, then $(q_{k,\ell} - q_{k,\ell'}) \in \mathbb{Q}_\mathrm{P}$ and so, by Lemma 2, 2^k divides $2\ell' - 1 - (2\ell - 1) = 2(\ell' - \ell)$, and hence 2^{k-1} divides $\ell - \ell'$. As $1 \leqslant \ell, \ell' \leqslant 2^{k-1}$, the only way this can occur is if $\ell = \ell'$. This shows that all 2^{k-1} cosets generated by $q_{k,\ell}$ are distinct.

Next, we show that these exhaust all possible cosets by elements of Q_{-k}. For this, given $q \in Q_{-k}$, we need to show that $q + \mathbb{Q}_\mathrm{P} = q_{k,\ell} + \mathbb{Q}_\mathrm{P}$ for some $\ell \in \{1, 2, \ldots 2^{k-1}\}$. As $q = n/(2^k m)$ for m and n odd, by Lemma 2, this amounts to showing $\nu(n - (2\ell - 1)m) \geqslant k$, or that 2^k divides $n + m - 2\ell m$. We may let $n + m = 2s$ ($s \in \mathbb{Z}$) and examine instead whether 2^{k-1} divides $s - \ell m$ for some $\ell \in \{1, \ldots, 2^{k-1}\}$.

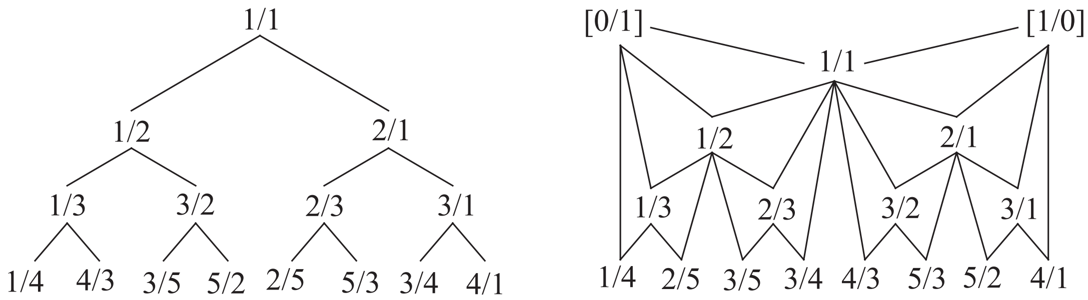

Figure 3. The initial rows of the Calkin-Wilf tree (left) and Stern-Brocot tree (right).

Notice that

$$\begin{aligned} s - \ell m \equiv s - \ell' m \pmod{2^{k-1}} &\iff 2^{k-1} \text{ divides } m(\ell - \ell') \\ &\overset{m \text{ odd}}{\iff} 2^{k-1} \text{ divides } (\ell - \ell') \iff \ell = \ell', \end{aligned}$$

as again $1 \leqslant \ell, \ell' \leqslant 2^{k-1}$. In particular, by the pigeonhole principle, all 2^{k-1} possible 2^{k-1}-remainders, including 0, are contributed by $s - \ell m$, $\ell = 1, \ldots, 2^{k-1}$. So 2^{k-1} divides $s - \ell m$ for some ℓ, and for this ℓ, we have $q \in q_{k,\ell} + \mathbb{Q}_{\mathrm{P}}$, as stated. ∎

Densities of the parity classes

There are many exhaustive sequences of rationals other than (1), one attractive option being the Calkin-Wilf tree [2]. The Calkin-Wilf tree is complete: it includes all positive rational numbers, and each such number occurs precisely once. The tree starts with the root value $1/1$, and everything springs from this root (see Figure 3, left panel). Each rational in the tree has two "children": for the entry m/n, the children are $m/(m+n)$ and $(m+n)/n$. The "left child" $m/(m+n)$ is always smaller than 1, while the "right child" $(m+n)/n$ is always greater than 1 (mnemonic: the *children* are "top over sum" and "sum over bottom").

The pattern of parity from one row of the Calkin-Wilf diagram to the next is simple. Denoting odd parity, even parity, and no parity by o, e, and n, respectively, the parity transfer rules are as follows.

$$\begin{array}{ccccccccc} & e & & & o & & & n & \\ \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\ e & & o & n & & e & o & & n \end{array} \tag{7}$$

We will now show that the elements of the Calkin-Wilf tree are remarkably regular, with the pattern (o, n, e) repeating interminably. Thus, the parity of any specific term in the tree can easily be deduced. We also prove that, with the ordering of $\mathbb{Q}$ corresponding to the Calkin-Wilf tree, the three parity classes all have the same density.

Theorem 1. *Let $\mathbb{Q}^+$ be ordered with the Calkin-Wilf tree. Then $\mathbb{Q}^+$ may be partitioned into three parity classes, $\mathbb{Q}^+_{\mathrm{E}}$, $\mathbb{Q}^+_{\mathrm{O}}$, and $\mathbb{Q}^+_{\mathrm{N}}$, each having asymptotic density $\frac{1}{3}$.*

Proof. The parity classes for the first few rows of the Calkin-Wilf tree are shown in Figure 4. Odd rows follow a pattern $(one)^k o$ for some $k \geqslant 0$, and even rows follow

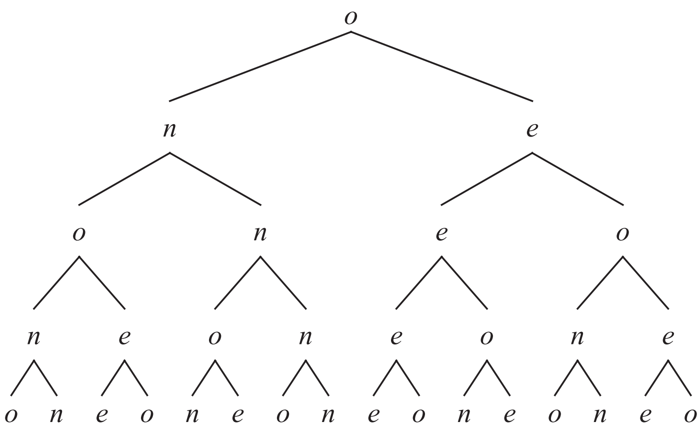

Figure 4. Parity classes of terms in the Calkin-Wilf tree.

a pattern $ne(one)^k$. This is clearly true for the first few rows. Using the transfer rules (7), and arguing inductively, it is clear that a row with pattern $(one)^k o$ is followed by a row with pattern $ne(one)^{2k}$. This, in turn, is followed by a row with pattern $(one)^{4k+1}o$. The full sequence begins

$$o \quad / \quad ne \quad / \quad (one)o \quad / \quad ne(one)^2 \quad / \quad (one)^5 o \quad / \quad ne(one)^{10} \quad / \quad \cdots .$$

We conclude that, if the entire tree is written row by row as a sequence, the parity follows an unvarying pattern, with *odd* followed by *none* followed by *even*. The parity of an element at any position N is immediately deduced from $m = N \pmod 3$. The pattern also implies that the three parity classes have equal densities. ■

The Calkin-Wilf tree enumerates the positive rationals $\mathbb{Q}^+$. This enumeration, which we write $\{q_n : n \in \mathbb{N}\}$, can be extended in a natural way to the full set of rationals: we enumerate $\mathbb{Q}$ by $\{0, q_1, -q_1, q_2, -q_2, \dots\}$. With this ordering, the rational numbers split into three parts, each of asymptotic density $\frac{1}{3}$.

To summarize, the parity classes of elements of the Calkin-Wilf tree follow a simple pattern if arranged in a single sequence: the pattern $\{o, n, e\}$ repeats indefinitely (see Figure 4). As a result, the densities of the parity classes in $\mathbb{Q}$ are all equal for this ordering:

$$\rho_{\mathbb{Q}}(\mathbb{Q}_\mathrm{E}) = \rho_{\mathbb{Q}}(\mathbb{Q}_\mathrm{O}) = \rho_{\mathbb{Q}}(\mathbb{Q}_\mathrm{N}) = \tfrac{1}{3}. \tag{8}$$

The Stern-Brocot tree [5, p. 116] is another ordering of $\mathbb{Q}$ very similar to the Calkin-Wilf tree. The numbers at each level are formed from the mediants of adjacent pairs of numbers above (Figure 3, right panel). The mediant of two (reduced) rationals, m_1/n_1 and m_2/n_2, is defined as $M(m_1/n_1, m_2/n_2) := (m_1 + m_2)/(n_1 + n_2)$. We note that the parity of the mediants of two numbers of different parity is the third parity:

$$M(e, o) = n, \qquad M(o, n) = e, \qquad M(n, e) = o. \tag{9}$$

We now show that, with the ordering of the Stern-Brocot tree, (8) holds true.

Theorem 2. *For the order of $\mathbb{Q}$ induced by the Stern-Brocot process, the asymptotic density of each parity class, $\mathbb{Q}_\mathrm{E}$, $\mathbb{Q}_\mathrm{O}$, and $\mathbb{Q}_\mathrm{N}$, is $\frac{1}{3}$.*

Proof. The Stern-Brocot tree is generated starting from level 0 with the boundary elements $\left[\frac{0}{1}\right]$ and $\left[\frac{1}{0}\right]$, representing 0 and ∞ and with parities $[e]$ and $[n]$. To get each subsequent level we add, between each pair of adjacent numbers, the mediant of that pair, retaining all numbers already generated. The results, for the first few levels, are shown in Figure 3 (right panel). The parity pattern for the first few levels is

$$[e]o[n] \quad / \quad [e]noe[n] \quad / \quad [e]oneoneo[n] \quad / \quad [e]noenoenoenoenoe[n].$$

Using the transfer rules (9), an odd row with parity $(eon)^K$ is followed by even row with parity $e(noe)^{2K-1}n$. This in turn is followed by an odd row with parity $(eon)^{4K-1}$. By an inductive argument, it follows that the parity pattern for an odd row k is $(eon)^K$, where $K = (2^k + 1)/3$ and, for an even row k, is $e(noe)^K n$, where $K = (2^\ell - 1)/3$. This implies that the asymptotic densities are equal for all three parity classes.

The Stern-Brocot tree enumerates the positive rationals $\mathbb{Q}^+$. This enumeration is easily extended to the full set of rationals, as was done above for the Calkin-Wilf tree. Then the rational numbers split into three parts, each of asymptotic density $\frac{1}{3}$. ■

The determination of the densities of parity classes for the ordering corresponding to the Farey sequences is left as a challenge for readers.

Conclusion

We have extended parity from the integers to the rational numbers. Three parity classes—even, odd, and "none"—were found. The even and odd rationals $\mathbb{Q}_E$ and $\mathbb{Q}_O$ follow the usual rules of parity. The union of these forms an additive subgroup $\mathbb{Q}_P$ of $\mathbb{Q}$.

Using the 2-adic valuation, we partitioned $\mathbb{Q}$ into subsets and found a chain of subgroups, each having a quotient group of cosets. We constructed a complete set of representatives for the cosets of $\mathbb{Q}_P$.

The Calkin-Wilf tree was found to have a remarkably simple parity pattern, with the sequence "odd/none/even" repeating indefinitely. Using the natural density, which provides a means of distinguishing the sizes of countably infinite sets, we showed that, with the Calkin-Wilf ordering, the three parity classes are equally dense in the rationals. The same conclusion holds for the Stern-Brocot tree.

Finally, we remark that, while this study discussed parity for the rational numbers, there may be potential for broad extensions and generalizations, to the p-adic numbers and to other number fields.

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Summary. We define an extension of parity from the integers to the rational numbers. Three parity classes are found—even, odd, and "none". Using the 2-adic valuation, we partition the rationals into subgroups with a rich algebraic structure.

The natural density provides a means of distinguishing the sizes of countably infinite sets. The Calkin-Wilf tree has a remarkably simple parity pattern, with the sequence "odd/none/even" repeating indefinitely. This pattern means that the three parity classes have equal natural density in the rationals. A similar result holds for the Stern-Brocot tree.

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