Computation of the Streamfunction and Velocity Potential and Reconstruction of the Wind Field

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ABSTRACT

A wind field given over a limited domain can be partitioned into nondivergent and irrotational components in an infinity of ways. A particular solution, selected by requiring the velocity potential to vanish on the boundary, has minimum divergent kinetic energy and is numerically easy to obtain.

The reconstruction of the wind field from the vorticity and divergence together with the boundary velocity is more difficult, since the potential equations are coupled by the boundary conditions. A numerical procedure is devised, which solves the two potential equations simultaneously, modifying both interior and boundary values in a converging iterative technique. The method is capable of reconstructing the wind field to any accuracy desired.

1. Introduction

The Helmholtz theorem allows us to partition the horizontal wind field **V** into nondivergent and irrotational components

$$\mathbf{V} = \mathbf{V}_\psi + \mathbf{V}_\chi = \mathbf{k} \times \nabla\psi + \nabla\chi, \tag{1}$$

where ψ is the streamfunction and χ the velocity potential. The vorticity ζ is defined as the vertical component of the curl of velocity, and from (1) we get

$$\nabla^2\psi = \zeta. \tag{2}$$

Similarly, taking the divergence of (1) we have

$$\nabla^2\chi = \delta, \tag{3}$$

where δ is the velocity divergence.

Two problems frequently arise in meteorological studies. First, given a windfield **V**, it is often of interest, for diagnostic or other purposes, to partition it into its nondivergent and irrotational components. We will call this the *partitioning problem.* Second, given the vorticity ζ and divergence δ, we wish to derive the velocity field. This we will call the *reconstruction problem.*

When the domain under consideration is the entire sphere, both problems are straightforward, and have unique solutions. However, when the domain is of limited extent the solution of the partitioning problem is not unique, and the reconstruction problem is complicated by boundary conditions which couple Eqs. (2) and (3). The streamfunction and velocity potential at the boundaries must satisfy (1), which means

$$\mathbf{s}\cdot\mathbf{V} = \frac{\partial\psi}{\partial n} + \frac{\partial\chi}{\partial s}, \tag{4}$$

$$\mathbf{n}\cdot\mathbf{V} = -\frac{\partial\psi}{\partial s} + \frac{\partial\chi}{\partial n}, \tag{5}$$

where **s** and **n** are tangential and normal unit vectors, respectively, and s and n are, respectively, distances along and normal to the boundary.

Several methods have been proposed for determining suitable, but not exact, boundary conditions in the case of a limited domain (Sangster, 1960; Hawkins and Rosenthal, 1965; Shukla and Saha, 1974; Schaefer and Doswell, 1979). Stephens and Johnson (1978) determined exact solutions for the streamfunction and velocity potential by assuming periodic boundary conditions and using Fourier transforms and consistent finitė difference approximations. However, their method cannot be applied in the case of spherical coördinates. They remark, "No means of specifying exact boundary condition values for a finite domain . . . have been available."

The method of Shukla and Saha (1974) was a clear improvement over previous methods. They used an iterative method, in which Eq. (2) and (3) were solved in alternate steps, using the most recent available boundary values. Examining the error in the reconstructed wind field, they showed that their method gives a smaller value of this error than any of the other methods tested. However, their method is "asymptotic" rather than "convergent"; there is a residual error which cannot be reduced by further iterations.

The method presented in this paper solves the reconstruction problem exactly, in the sense that the wind field can be recovered to any degree of accuracy desired. The method differs from that of Shukla and Saha in that (2) and (3) are solved simultaneously rather than

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sequentially. A relaxation technique (SOR) is used, in which the boundary values of ψ and χ are modified after each iteration. In the specific case considered, 12 iterations were needed to reduce the maximum (absolute) residue by a factor of 10. The method is more economical than that of Shukla and Saha, which requires repeated solutions of the Poisson Eqs. (2) and (3); in the present case these equations are solved only once. The method uses consistent finite difference approximations for the potential equations.

2. The partitioning problem: nonuniqueness of the solution

If the wind components (u, v) are known over the sphere, they may be partitioned uniquely into nondivergent and irrotational parts: we evaluate the vorticity ζ and divergence δ by differentiation, and solve two Poisson equations:

$$\nabla^2\psi = \zeta, \tag{6}$$

$$\nabla^2\chi = \delta, \tag{7}$$

for the streamfunction ψ and velocity potential χ. There are no boundary conditions, but the solutions are required to be regular on the sphere. These solutions are unique up to additive constants; this is so because the solutions of the corresponding homogeneous equations (Laplace equations) are harmonic functions which cannot have extrema in the interior of the domain and are therefore constants. Thus, the partitioning of the wind field into nondivergent and irrotational components is unique.

For a limited area the splitting of the wind field is no longer unique. It was stated by Miyakoda (1960) that in this case the distributions of ψ and χ have no physical significance of themselves, but only insofar as they are used in (1). We examine this question in detail below.

From a wind field (u, v) in a region Ω, bounded by a curve Γ, we can derive the vorticity ζ and divergence δ:

$$\zeta = \frac{1}{\sigma}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial(\sigma u)}{\partial\theta}\right], \tag{8}$$

$$\delta = \frac{1}{\sigma}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial(\sigma v)}{\partial\theta}\right]. \tag{9}$$

Here, λ and θ are longitude and latitude, $\sigma = \cos\theta$, and the radius of the sphere is taken as unity. The streamfunction and velocity potential are obtained by solving (6) and (7) subject to the boundary conditions on Γ:

$$-\frac{\partial\psi}{\partial\theta} + \frac{1}{\sigma}\frac{\partial\chi}{\partial\lambda} = u, \tag{10}$$

$$\frac{1}{\sigma}\frac{\partial\psi}{\partial\lambda} + \frac{\partial\chi}{\partial\theta} = v. \tag{11}$$

Since these conditions involve both variables, the system is coupled and the two equations must be solved together. The system has a unique solution if and only if the corresponding homogeneous system has only the trivial solution where ψ and χ are constants; however, this is not the case. Consider the example:

$$\left.\begin{aligned}\psi &= A\log(\sec\theta + \tan\theta) + B\lambda \\ \chi &= -B\log(\sec\theta + \tan\theta) + A\lambda\end{aligned}\right\}, \tag{12}$$

where A and B are arbitrary constants. By direct substitution we see that these functions satisfy the homogeneous system. The corresponding wind fields are derived from (1) and are seen to *vanish*. However, their nondivergent and irrotational parts are as follows:

$$\left.\begin{aligned}u_\psi &= -\frac{\partial\psi}{\partial\theta} = -\frac{A}{\sigma} \\ v_\psi &= \frac{1}{\sigma}\frac{\partial\psi}{\partial\lambda} = \frac{B}{\sigma} \\ u_\chi &= \frac{1}{\sigma}\frac{\partial\chi}{\partial\lambda} = \frac{A}{\sigma} \\ v_\chi &= \frac{\partial\chi}{\partial\theta} = -\frac{B}{\sigma}\end{aligned}\right\}. \tag{13}$$

The nondivergent part (u_ψ, v_ψ), which is also irrotational, is a flow with constant direction and speed varying inversely as $\cos\theta$. The irrotational part (u_χ, v_χ), also nondivergent, is simply the reverse flow.

It is now clear that the partitioning of the wind field cannot be unique. Given any separation into nondivergent and irrotational components:

$$u = u_\psi + u_\chi,$$

$$v = v_\psi + v_\chi,$$

we can add the solution (13), which changes the components arbitrarily without altering the total flow. Furthermore, (12) and (13) are not the most general solution of the homogeneous problem.

From the infinitude of possible partitions of the wind field we can select a particular solution in the following way. Let (ψ, χ) be any such solution. We write

$$\psi = \psi_0 + \hat{\psi},$$

$$\chi = \chi_0 + \hat{\chi},$$

where $\hat{\psi}$ and $\hat{\chi}$ *vanish* on the boundary Γ and satisfy the Poisson equations (6) and (7) in the interior, Ω. (Note that $\hat{\psi}$ and $\hat{\chi}$ are uniquely determined by these conditions.) Then, the remaining parts (ψ_0, χ_0) are equal to the original solution (ψ, χ) on the boundary, and satisfy Laplace equations

$$\nabla^2\psi_0 = 0, \quad \nabla^2\chi_0 = 0$$

in the interior. (They are again uniquely determined once ψ and χ have been specified.) These functions determine that part of the flow which is *both* nondivergent *and* irrotational. (On the sphere this part necessarily vanishes, since potential flow must have a singularity somewhere and thus is a physical flow only over a bounded sector of the sphere.) We may now partition the wind field as follows:

$$\mathbf{V} = \mathbf{V}_0 + \hat{\mathbf{V}}_\psi + \hat{\mathbf{V}}_\chi$$

and the nondivergent *and* irrotational component, $\mathbf{V}_0$, may be divided arbitrarily between the rotational and divergent components. To *render the partitioning unique,* we include $\mathbf{V}_0$ with the rotational component:

$$\mathbf{V}_\psi = \mathbf{V}_0 + \hat{\mathbf{V}}_\psi, \quad \mathbf{V}_\chi = \hat{\mathbf{V}}_\chi.$$

The partitioning problem is now solved very simply. Given the (u, v) field, we derive the divergence δ and solve the Poisson equation (7) *with zero boundary conditions;* this determines $\hat{\chi}$ uniquely. We may then derive the divergent wind $\mathbf{V}_\chi = \nabla\hat{\chi}$, and the residue $(\mathbf{V} - \mathbf{V}_\chi)$ is the rotational component. We do not need to solve explicitly for (ψ, χ) or (ψ_0, χ_0). (Sangster [1960] has argued that a zero boundary condition on χ is appropriate for minimizing the kinetic energy of the divergent wind and maximizing that of the rotational flow; this is, in general, a meteorologically reasonable criterion.)

Consider now the *reconstruction problem.* If the velocity field is given only on the boundary, together with the vorticity and divergence in the interior, then we determine the velocity everywhere by solving (6) and (7) in Ω subject to (10) and (11) on Γ, and using (1). Any solution will do, since all yield the same wind field. However, the rotational and divergent components of the flow can be specified uniquely only after we determine $\hat{\chi}$ by solving a further Poisson equation.

Finally, let us again emphasize that in all cases the total wind field is well determined by the given data. It is the partitioning into nondivergent and irrotational components that is nonunique without further conditions.

3. Numerical solution of the reconstruction problem

We have seen that the problem of recovering the wind field from the vorticity and divergence involves the simultaneous solution of the two Poisson Eqs. (2) and (3) which are coupled by the boundary conditions (4) and (5). In this section we describe a numerical method for solving these equations together, and we show by an example that the method is capable of recovering the wind field to any desired degree of accuracy. The method is an iterative procedure in which both the interior and boundary values of ψ and χ are relaxed towards balance with the interior and boundary forcing.

a. Details of the method

We consider a regular longitude–latitude grid $\lambda_i = i\Delta\lambda$, $\theta_j = j\Delta\theta$, with variables distributed as in Fig. 1. (It is essentially an Arakawa C-grid.) We approximate

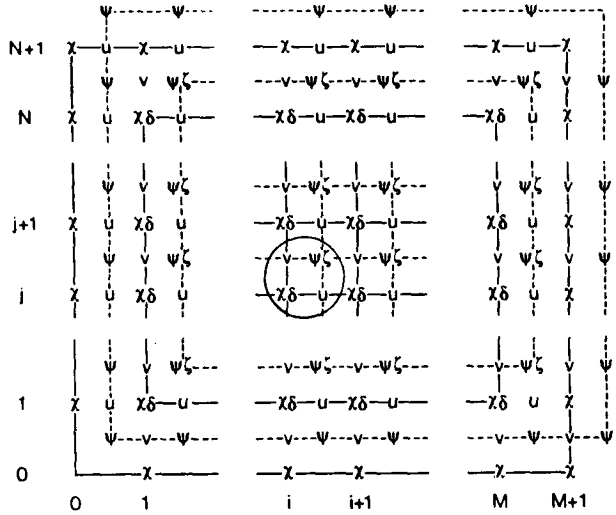

FIG. 1. Disposition of the variables on the discrete grid. Values of variables within the circle are indicated by subscripts i, j.

derivatives by centered differences on this grid. When Eq. (1) is split into components and discretized we get

$$u_{ij} = -(1/\Delta\theta)(\psi_{ij} - \psi_{ij-1}) + (1/\sigma_j\Delta\lambda)(\chi_{i+1j} - \chi_{ij}), \tag{14}$$

$$v_{ij} = (1/\sigma_{j+1/2}\Delta\lambda)(\psi_{ij} - \psi_{i-1j}) + (1/\Delta\theta)(\chi_{ij+1} - \chi_{ij}), \tag{15}$$

where $\sigma_j = \cos j\Delta\theta$, $u_{ij} = u[(i + ½)\Delta\lambda, j\Delta\theta]$, etc. The vorticity and divergence are related to the winds through (8) and (9), which become

$$\zeta_{ij} = (1/\sigma_{j+1/2})[(1/\Delta\lambda)(v_{i+1j} - v_{ij}) - (1/\Delta\theta)(\sigma_{j+1}u_{ij+1} - \sigma_j u_{ij})], \tag{16}$$

$$\delta_{ij} = (1/\sigma_j)[(1/\Delta\lambda)(u_{ij} - u_{i-1j}) + (1/\Delta\theta)(\sigma_{j+1/2}v_{ij} - \sigma_{j-1/2}v_{ij-1})]. \tag{17}$$

Now using (14) and (15) in (16) and (17) we get the finite difference analogues of the Poisson equations (2) and (3):

$$\nabla_d^2\psi_{ij} = \zeta_{ij}, \tag{18}$$

$$\nabla_d^2\chi_{ij} = \delta_{ij}, \tag{19}$$

where ∇_d^2 is the usual five-point discrete Laplacian operator. When these equations are applied on the $M \times N$ interior grid points where ζ and δ are defined (see Fig. 1), boundary values of ψ and χ occur with $i = 0$, $M + 1$ or $j = 0$, $N + 1$. Since the boundary values of u and v are known, we may solve (14) or (15) for these values. Equation (14) is used to evaluate

$$\psi_{i0}, \psi_{iN+1}, \chi_{0j}, \chi_{M+1j}$$

and (15) is used for

$$\psi_{0j}, \psi_{M+1j}, \chi_{i0} \quad \text{and} \quad \chi_{iN+1}.$$

For example,

$$\psi_{i0} = \psi_{i1} - (\Delta\theta/\sigma_1\Delta\lambda)(\chi_{i+1\,1} - \chi_{i1}) + \Delta\theta u_{i1},$$

$$\psi_{i\,N+1} = \psi_{iN} + (\Delta\theta/\sigma_{N+1}\Delta\lambda)(\chi_{i+1\,N+1} - \chi_{i\,N+1}) - \Delta\theta u_{i\,N+1}.$$

In all cases, the most recent values available are used to evaluate the right-hand sides during the iteration process. Thus we see that both the boundary and interior values of ψ and χ change during the solution process; only the boundary velocities and interior forcing (ζ, δ) are unchanging. The quantities ψ_{00} and $\chi_{M+1\,N+1}$ are set arbitrarily to zero.

The system of equations (18), (19) is now written in the form

$$\nabla_d^2\psi_{ij} = \zeta_{ij} - P_{ij}, \qquad (20)$$

$$\nabla_d^2\chi_{ij} = \delta_{ij} - Q_{ij}, \qquad (21)$$

where ∇_d^2 denotes the interior terms of the discrete Laplacian, and the boundary terms of ψ and χ are written on the right-hand side in P_{ij} and Q_{ij}.

The system (20), (21) is now solved simultaneously using a successive overrelation (SOR) technique. We start from some first guess, for example $\psi \equiv \chi \equiv 0$. A single SOR iteration of (20) is made during which the values of ψ are updated. There follows an iteration through (21), using the most recent values of ψ to evaluate the boundary values of χ from (14) or (15). The ψ-iteration is repeated, using updated χ-values at the boundaries, and so on. All the ψ and χ values are changing, and thus the right-hand forcing terms in (20), (21) also change at each iteration.

The characteristic which distinguishes the present solution method from previously proposed methods (e.g., that of Shukla and Saha) is the modification of the boundary values of χ and ψ *during* the actual solution procedure. In other methods, these values are fixed, so that the solution has to be repeated iteratively. Apparently, it is this characteristic of the method which leads to its good convergence properties, and its ability to reconstruct the wind field exactly.

If equations (20) and (21) are combined to form a single equation, it may be possible to apply a direct solution method. However, it is not clear whether the computational expense would be reduced. Since our primary interest is in reconstructing the wind field *accurately*, we have chosen the more simple SOR method.

b. A numerical example

The above algorithm is now used to solve for the wind components on a grid with $(M \times N) = (21 \times 20)$ internal grid points having a mesh spacing $(\Delta\lambda, \Delta\theta) = (2°, 1°)$, and covering approximately the area between 45° and 65°N and 20°W and 20°E.

The wind components u and v are shown in Fig. 2, and represent the circulation around a depression near the centre of the region.

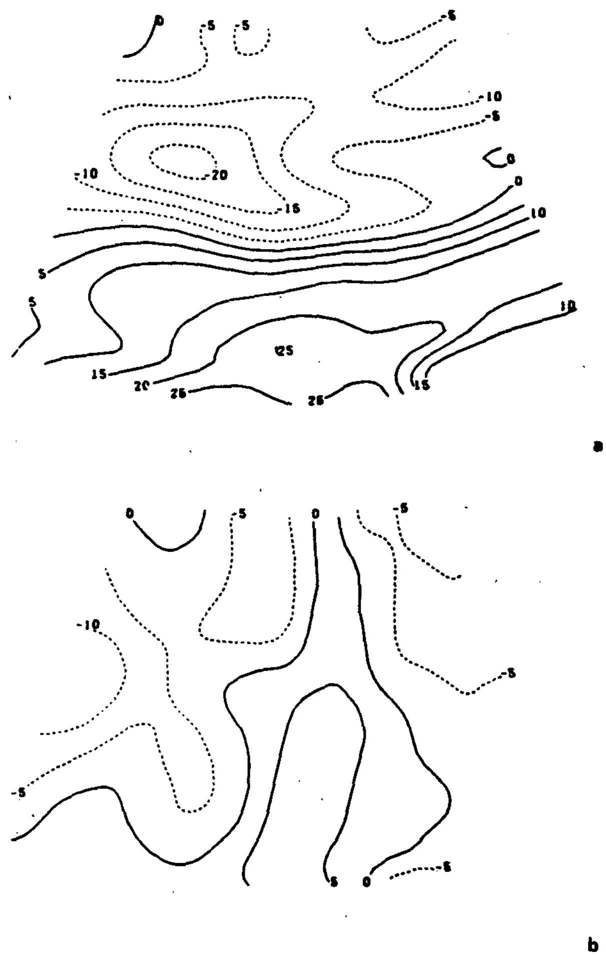

FIG. 2. Initial wind field: a) zonal component u, and b) meridional component v. (Units are m s^{-1}.)

From these winds we derive vorticity and divergence fields over the interior of the region. These latter forcing fields are then used, together with the winds on the boundary, to derive the streamfunction and velocity potential using the numerical technique described above. The fields of ψ and χ after 60 iterations are shown in Figs. 3 and 4, respectively.

The solution process was started using zero values for ψ and χ. The wind fields corresponding to ψ and χ were evaluated at several stages during the iteration process, using (14) and (15), and compared with the original wind fields. The maximum (absolute) differences in each velocity component are shown in Table 1. We can see from these results that the reconstructed wind (from ψ and χ) is converging satisfactorily towards the original wind, and that the differences are negligible after about 30 iterations.

4. Conclusions

The problem of *partitioning* the wind field into rotational and divergent components has been examined.

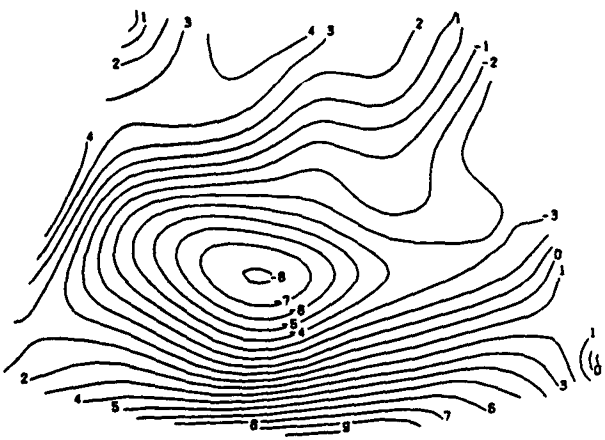

FIG. 3. Streamfunction ψ computed from the wind field in Fig. 2 after 60 iterations. (Units are 10^6 m^2 s^{-1}.)

On a limited domain the solution is nonunique; a particular solution was chosen by requiring the velocity potential to vanish on the boundary. This solution has minimum divergent kinetic energy (Sangster, 1960) and is obtained by the numerically straightforward solution of a Poisson equation with zero boundary conditions.

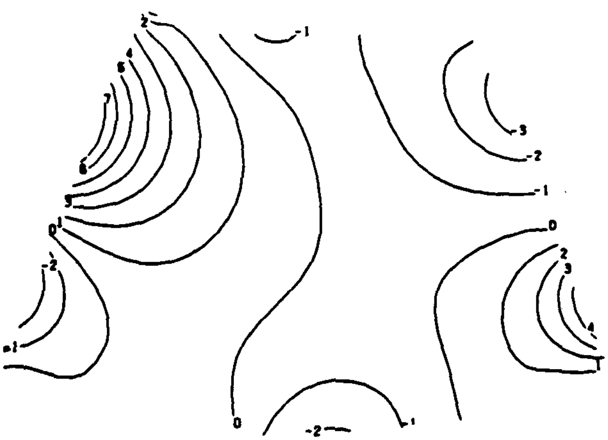

FIG. 4. As in Fig. 3 but for the velocity potential χ.

TABLE 1. Maximum absolute differences (m s^{-1}) between the original and reconstructed wind components for 60 iterations of the solution technique. (The overrelaxation parameter was chosen as 1.80.)

Iterations	$\lvert\Delta u\rvert_{max}$	$\lvert\Delta v\rvert_{max}$
0	27.	14.
12	2.8	2.0
24	4.2×10^{-1}	4.9×10^{-1}
36	5.9×10^{-2}	5.5×10^{-2}
48	8.2×10^{-4}	1.2×10^{-3}
60	1.6×10^{-4}	1.1×10^{-4}

The more difficult problem of *reconstructing* the wind field from the vorticity and divergence has been considered, and an iterative technique for its solution has been devised. This technique relaxes both interior and boundary values of the streamfunction and velocity potential towards balance with the forcing functions. Its effectiveness has been shown by application to a particular wind field. In contrast to previously proposed methods known to us (see, e.g., Shukla and Saha, 1974), the solution converges towards the original wind field, which may be recovered to any degree of accuracy required.

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