INTRODUCTION TO INITIALIZATION

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1. Introduction

The spectrum of atmospheric motions is vast, encompassing phenomena having periods ranging from seconds to millennia. The motions of interest to the forecaster have timescales greater than a day, but the mathematical models used for numerical prediction describe a broader span of dynamical features than those of direct concern. For many purposes these higher frequency components can be regarded as *noise* contaminating the motions of meteorological interest. The elimination of this noise is achieved by adjustment of the initial fields, a process called *initialization.* In this chapter, the fundamental equations are examined and the causes of spurious oscillations are elucidated. The history of methods of eliminating high-frequency noise is recounted and various initialization methods are described. The normal mode initialization method is described, and illustrated by application to a simple mechanical system, the swinging spring.

1.1. RICHARDSON'S FORECAST

The story of Lewis Fry Richardson's forecast, made about eighty years ago, is well known. Richardson forecast the change in surface pressure at a point in central Europe, using the mathematical equations. He described his methods and results in his book *Weather Prediction by Numerical Process* (Richardson, 1922). His results implied a change in surface pressure of 145 hPa in 6 hours. As Sir Napier Shaw remarked, "the wildest guess ... would not have been wider of the mark ...". Yet, Richardson claimed that his forecast was "... a fairly correct deduction from a somewhat unnatural initial distribution"; he ascribed the unrealistic value of pressure tendency to errors in the observed winds, leading to a spuriously large value of the calculated divergence. This large tendency reflects the fact that the atmosphere can support motions with a great range of timescales.

1.2. THE SPECTRUM OF ATMOSPHERIC MOTIONS

The natural oscillations of the atmosphere fall into two groups (see, *e.g.*, Kasahara, 1976). The solutions of meteorological interest have low frequencies and are close to geostrophic balance. They are called rotational or vortical modes, since their vorticity is greater than their divergence; if divergence is ignored, these modes reduce to the Rossby-Haurwitz waves. There are also very fast gravity-inertia wave solutions, with phase speeds of hundreds of metres per second and large divergence. For typical conditions of large scale atmospheric flow (when the Rossby and Froude numbers are small) the two types of motion are clearly separated and interactions between them are weak. The high frequency gravity-inertia waves may be locally significant in the vicinity of steep orography, where there is strong thermal forcing or where very rapid changes are occurring; but overall they are of minor importance and may be regarded as undesirable noise.

1.3. THE PROBLEM OF INITIALIZATION.

A subtle and delicate state of balance exists in the atmosphere between the wind and pressure fields, ensuring that the fast gravity waves have much smaller amplitude than the slow rotational part of the flow. Observations show that the pressure and wind fields in regions not too near the equator are close to a state of geostrophic balance and the flow is quasi-nondivergent. The bulk of the energy is contained in the slow rotational motions and the amplitude of the high frequency components is small. The existence of this geostrophic balance is a perennial source of interest; it is a consequence of the forcing mechanisms and dominant modes of hydrodynamic instability and of the manner in which energy is dispersed and dissipated in the atmosphere. For a review of balanced flow, see McIntyre (2003). The gravity-inertia waves are instrumental in the process by which the balance is maintained, but the nature of the sources of energy ensures that the low frequency components predominate in the large scale flow. The atmospheric balance is subtle, and difficult to specify precisely. It is *delicate* in that minor perturbations may disrupt it but *robust* in that local imbalance tends to be rapidly removed through radiation of gravity-inertia waves in a process known as geostrophic adjustment.

When the primitive equations are used for numerical prediction the forecast may contain spurious large amplitude high frequency oscillations. These result from anomalously large gravity-inertia waves which occur because the balance between the mass and velocity fields is not reflected faithfully in the analysed fields. High frequency oscillations of large amplitude are engendered, and these may persist for a considerable time unless strong dissipative processes are incorporated in the forecast model.

One of the long-standing problems in numerical weather prediction has been to overcome the problems associated with high frequency motions. This is achieved by the process known as *initialization*, the principal aim of which is to define the initial fields in such a way that the gravity inertia waves remain small throughout the forecast. If the fields are not initialized the spurious oscillations which occur in the forecast can lead to various problems. In particular, new observations are checked for accuracy against a short-range forecast. If this forecast is noisy, good observations may be rejected or erroneous ones accepted. Thus, *initialization is essential for satisfactory data assimilation.* Another problem occurs with precipitation forecasting. A noisy forecast has unrealistically large vertical velocity. This interacts with the humidity field to give hopelessly inaccurate rainfall patterns. To avoid this *spin-up*, we must control the gravity wave oscillations.

2. Scale-analysis of the Shallow Water Equations.

Considerable insight into the problem of initialization is achieved by consideration of the scale properties of the linear shallow water equations. We introduce characteristic scales for the dependent variables, and examine the relative sizes of the terms in the equations. Let $L = 10^6\,\mathrm{m}$ represent the length scale and $V = 10\,\mathrm{m\,s^{-1}}$ the velocity scale. An advective time-scale $T = L/V = 10^5\,\mathrm{s}$ is assumed. P represents the scale of pressure variations. For simplicity we take $f = 10^{-4}\,\mathrm{s^{-1}}$ and density $\rho_0 = 1\,\mathrm{kg\,m^{-3}}$ to be constants. The linear rotational shallow water equations are:

$$\underbrace{\frac{\partial u}{\partial t}}_{V/T} - \underbrace{fv}_{fV} + \underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial x}}_{P/L} = 0 \qquad (2.1)$$

$$\underbrace{\frac{\partial v}{\partial t}}_{V/T} + \underbrace{fu}_{fV} + \underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial y}}_{P/L} = 0 \qquad (2.2)$$

$$\underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial t}}_{P/T} + \underbrace{gH\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]}_{gHV/L} = 0 \qquad (2.3)$$

The scale of each term in the equations is indicated. The ratio of the velocity tendencies to the Coriolis terms is the Rossby number,

$$\mathrm{Ro} \equiv \frac{1}{fT} = \frac{V}{fL} = 10^{-1}\,,$$

a small parameter. For balance in the momentum equations, we must have $P = LfV = 10^3\,\mathrm{Pa}$. Then the sizes of the terms are as indicated here:

$$\underbrace{\frac{\partial u}{\partial t}}_{10^{-4}} - \underbrace{fv}_{10^{-3}} + \underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial x}}_{10^{-3}} = 0 \qquad (2.4)$$

$$\underbrace{\frac{\partial v}{\partial t}}_{10^{-4}} \quad + \quad \underbrace{fu}_{10^{-3}} \quad + \quad \underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial y}}_{10^{-3}} \quad = \quad 0\,. \tag{2.5}$$

To the lowest order of approximation, the tendency terms are negligible and there is geostrophic balance between the Coriolis and pressure terms.

2.1. SCALING THE DIVERGENCE

The vorticity is the same scale as each of its components:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \sim \frac{V}{L} = 10^{-5}\,\mathrm{s}^{-1}\,.$$

Due to the cancellation between the two terms in the divergence, one might expect it to scale an order of magnitude smaller than each of its terms:

$$\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \sim \mathrm{Ro}\frac{V}{L} = 10^{-6}\,\mathrm{s}^{-1}$$

This is generally appropriate but here we are considering barotropic motions of large vertical scale. If we assume this magnitude for the divergence, and take $g = 10\,\mathrm{m\,s^{-2}}$ and $H = 10^4$ m (the scale height of the atmosphere is approximately 10 km), the terms of the continuity equation scale as follows:

$$\underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial t}}_{10^{-2}} \quad + \quad \underbrace{gH\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]}_{10^{-1}} \quad = \quad 0\,, \tag{2.6}$$

which is *impossible*, as there is nothing to balance the second term. We recall that the divergence term arises through vertical integration:

$$g\int \delta\, dz \quad \approx \quad gH\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]\,.$$

There is a strong tendency in the atmosphere towards cancellation between convergence at low levels and divergence at higher levels and vice-versa, called the Dines mechanism. Thus, we assume

$$\int \delta\, dz \sim \mathrm{Ro}\,\delta H\,, \qquad \text{so that} \qquad g\int \delta\, dz \sim \mathrm{Ro}^2 gH\frac{V}{L} = 10^{-2}\,.$$

The terms of the continuity equation are now brought into balance.

2.2. THE EFFECT OF DATA ERRORS

Suppose there is a 10% error Δv in the v-component of the wind observation at a point. The scales of the terms are as before:

$$\underbrace{\frac{\partial u}{\partial t}}_{10^{-4}} \quad - \quad \underbrace{f(v + \Delta v)}_{10^{-3}} \quad + \quad \underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial x}}_{10^{-3}} \quad = \quad 0 \tag{2.7}$$

However, the error in the tendency is $\Delta(\partial u/\partial t) \sim f\Delta v \sim 10^{-4}$, comparable in size to the tendency itself: the signal-to-noise ratio is 1. The forecast may be qualitatively reasonable, but will be quantitatively invalid.

A similar conclusion is reached for a 10% error in the pressure gradient. However, if the spatial scale Δx of the pressure error is small (say, $\Delta x \sim L/10$) the error in its gradient is correspondingly large:

$$\frac{\partial p}{\partial x} \sim \frac{P}{L}, \qquad \text{but} \qquad \Delta\frac{\partial p}{\partial x} \sim \frac{\Delta p}{\Delta x} \sim \frac{P}{L} \sim \frac{\partial p}{\partial x},$$

so that the error in the wind tendency is now

$$\Delta\frac{\partial u}{\partial t} \sim \frac{1}{\rho_0}\frac{\partial p}{\partial x} \sim 10^{-3} \gg \frac{\partial u}{\partial t}.$$

The forecast will be qualitatively incorrect, indeed unreasonable.

Now consider the continuity equation. The pressure tendency has scale

$$\frac{\partial p}{\partial t} \sim 10^{-2}\,\mathrm{Pa\,s^{-1}} \approx 1\,\mathrm{hPa\ in\ 3\ hours}.$$

If there is a 10% error in the wind, the resulting error in divergence is $\Delta\delta \sim \Delta v/L \sim 10^{-6}$. The error is larger than the divergence itself! As a result, the pressure tendency is unrealistic. Worse still, if the wind error is of small spatial scale, the divergence error is correspondingly greater:

$$\Delta\delta \sim \Delta\frac{\partial v}{\partial x} \sim \frac{\Delta v}{\Delta x} \sim \frac{V}{L} \sim 10^{-5} \sim 10^2\delta.$$

Clearly, this implies a pressure tendency *two orders of magnitude larger* than the correct value. Instead of the value $\partial p/\partial t \sim 1$ hPa in 3 hours we get a change of order 100 hPa in 3 hours. This is strikingly reminiscent of Richardson's result.

3. Early Initialization Methods

3.1. THE FILTERED EQUATIONS

The first computer forecast was made in 1950 by Charney, Fjørtoft and Von Neumann. In order to avoid Richardson's error, they modified the prediction equations in such a way as to eliminate the high frequency solutions. This process is known as filtering. The basic filtered system is the set of quasi-geostrophic equations. These equations were used in operational forecasting for a number of years. However, they involve approximations which are not always valid, and this can result in poor forecasts. A more accurate filtering of the primitive equations leads to the *balance equations*. This system is more complicated to solve than the quasi-geostrophic system, and has not been widely used.

3.2. STATIC INITIALIZATION

Hinkelmann (1951) investigated the problem of noise in numerical integrations and concluded that if the initial winds were geostrophic, high frequency oscillations would occur but would remain small in amplitude. He later succeeded in integrating the primitive equations, using a very short timestep, with geostrophic initial winds (Hinkelmann, 1959). Forecasts made with the primitive equations were soon shown to be clearly superior to those using the quasi-geostrophic system. However, the use of geostrophic initial winds had a huge disadvantage: the valuable information contained in the observations of the wind field was completely ignored. Moreover, the remaining noise level is not tolerable in practice. Charney (1955) proposed that a better estimate of the initial wind field could be obtained by using the nonlinear balance equation. This equation — part of the balance system — is a diagnostic relationship between the pressure and wind fields. It implies that the wind is nondivergent. It was later argued by Phillips (1960) that a further improvement would result if the divergence of the initial field were set equal to that implied by quasi-geostrophic theory. Each of these steps represented some progress, but the noise problem still remained essentially unsolved.

3.3. DYNAMIC INITIALIZATION

Another approach, called dynamic initialization, uses the forecast model itself to define the initial fields. The dissipative processes in the model can damp out high frequency noise as the forecast procedes. We integrate the model first forward and then backward in time, keeping the dissipation active all the time. We repeat this forward-backward cycle many times until we finally obtain fields, valid at the initial time, from which the high frequency components have been damped out. The forecast starting from these fields is noise-free. However, the procedure is expensive in computer time, and damps the meteorologically significant motions as well as the gravity waves, so it is no longer popular. Digital filtering initialization, described in another chapter of this book, is essentially a refinement of dynamic initialization. Because it used a highly selective filtering technique, is is computationally more efficient than the older method.

3.4. VARIATIONAL INITIALIZATION

An elegant initialization method based on the calculus of variations was introduced by Sasaki (1958). We consider the simplest case: given an analysis of the mass and wind fields, how can they be modified so as to impose geostrophic balance? This problem can be be formulated as the minimization of an integral representing the deviation of the resulting fields from balance. The variation of the integral leads to the Euler-Lagrange equations, which yield diagnostic relationships for the new mass and wind fields in terms of the incoming analysis. Although the method was not widely used, the variational method is now at the centre of modern data assimilation practice.

4. Atmospheric Normal Mode Oscillations

The solutions of the model equations can be separated, by a process of spectral analysis, into two sets of components or linear normal modes, slow rotational components or Rossby modes, and high frequency gravity modes. We assume that the amplitude of the motion is so small that all nonlinear terms can be neglected. The horizontal structure is then governed by a system equivalent to the linear shallow water equations which describe the small-amplitude motions of a shallow layer of incompressible fluid. These equations were first derived by Laplace in his discussion of tides in the atmosphere and ocean, and are called the Laplace Tidal Equations. The simplest means of deriving the linear shallow water equations from the primitive equations is to assume that the vertical velocity vanishes identically.[1]

4.1. THE LAPLACE TIDAL EQUATIONS

Let us assume that the motions under consideration can be described as small perturbations about a state of rest, in which the temperature is a constant, T_0, and the pressure $\bar{p}(z)$ and density $\bar{\rho}(z)$ vary only with height. The basic state variables satisfy the gas law and are in hydrostatic balance: $\bar{p} = \mathcal{R}\bar{\rho}T_0$ and $d\bar{p}/dz = -g\bar{\rho}$. The variations of mean pressure and density follow immediately:

$$\bar{p}(z) = p_0 \exp(-z/H)\,, \qquad \bar{\rho}(z) = \rho_0 \exp(-z/H)\,,$$

where $H = p_0/g\rho_0 = \mathcal{R}T_0/g$ is the scale-height of the atmosphere. We consider only motions for which the vertical component of velocity vanishes identically, $w \equiv 0$. Let u, v, p and ρ denote variations about the basic state, each of these being a small quantity. The horizontal momentum, continuity and thermodynamic equations become

$$\frac{\partial \bar{\rho}u}{\partial t} - f\bar{\rho}v + \frac{\partial p}{\partial x} = 0 \tag{4.1}$$

$$\frac{\partial \bar{\rho}v}{\partial t} + f\bar{\rho}u + \frac{\partial p}{\partial y} = 0 \tag{4.2}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\bar{\rho}\mathbf{V} = 0 \tag{4.3}$$

$$\frac{1}{\gamma\bar{p}}\frac{\partial p}{\partial t} - \frac{1}{\bar{\rho}}\frac{\partial \rho}{\partial t} = 0 \tag{4.4}$$

[1]This assumption precludes the possibility of studying internal modes for which $w \neq 0$. A more general derivation, based on a separation of the horizontal and vertical dependencies of the variables, is presented in Daley, (1991, Ch. 9).

Density can be eliminated from the continuity equation (4.3) by means of the thermodynamic equation (4.4). Now let us assume that the horizontal and vertical dependencies of the perturbation quantities are separable:

$$\left\{ \begin{array}{c} \bar{\rho}u \\ \bar{\rho}v \\ p \end{array} \right\} = \left\{ \begin{array}{c} U(x,y,t) \\ V(x,y,t) \\ P(x,y,t) \end{array} \right\} Z(z) \,. \tag{4.5}$$

The momentum and continuity equations can then be written

$$\frac{\partial U}{\partial t} - fV + \frac{\partial P}{\partial x} = 0 \tag{4.6}$$

$$\frac{\partial V}{\partial t} + fU + \frac{\partial P}{\partial y} = 0 \tag{4.7}$$

$$\frac{\partial P}{\partial t} + (gh)\nabla \cdot \mathbf{V} = 0 \tag{4.8}$$

where $\mathbf{V} = (U, V)$ is the momentum vector and $h = \gamma H = \gamma \mathcal{R} T_0 / g$. This is a set of three equations for the three dependent variables U, V, and P. They are mathematically isomorphic to the Laplace tidal equations with a mean depth h. The quantity h is called the equivalent depth. There is no dependence in this system on the vertical coordinate z.

The vertical structure follows from the hydrostatic equation, together with the relationship $p = (\gamma g H)\rho$ implied by the thermodynamic equation. It is determined by

$$\frac{dZ}{dz} + \frac{Z}{\gamma H} = 0 \,, \tag{4.9}$$

the solution of which is $Z = Z_0 \exp(-z/\gamma H)$, where Z_0 is the amplitude at $z = 0$. If we set $Z_0 = 1$, then U, V and P give the momentum and pressure fields at the earth's surface. These variables all decay exponentially with height. It follows from (4.5) that u and v actually increase with height as $\exp(\kappa z/H)$, but the kinetic energy decays. Solutions with more general vertical structures, and with non-vanishing vertical velocity, are discussed in Daley, (1991, Ch. 9).

4.2. VORTICITY AND DIVERGENCE

We examine the solutions of the Laplace Tidal Equations in some enlightening limiting cases. Holton (1975) gives a more extensive analysis, including treatments of the equatorial and mid-latitude β-plane approximations. By means of the Helmholtz Theorem, a general horizontal wind field $\mathbf{V}$ may be partitioned into rotational and divergent components

$$\mathbf{V} = \mathbf{V}_\psi + \mathbf{V}_\chi = \mathbf{k} \times \nabla\psi + \nabla\chi \,.$$

The stream function ψ and velocity potential χ are related to the vorticity and divergence by the Poisson equations $\nabla^2\psi = \zeta$ and $\nabla^2\chi = \delta$.

It is straightforward to derive equations for the vorticity and divergence tendencies. Together with the continuity equation, they are

$$\frac{\partial \zeta}{\partial t} + f\delta + \beta v = 0 \tag{4.10}$$

$$\frac{\partial \delta}{\partial t} - f\zeta + \beta u + \nabla^2 P = 0 \tag{4.11}$$

$$\frac{\partial P}{\partial t} + gh\delta = 0\,. \tag{4.12}$$

These equations are completely equivalent to (4.6)–(4.8); no additional approximations have yet been made. However, the vorticity and divergence forms enable us to examine various simple approximate solutions.

4.3. ROSSBY-HAURWITZ MODES

If we suppose that the solution is quasi-nondivergent, *i.e.*, we assume $|\delta| \ll |\zeta|$, the wind is given approximately in terms of the stream function $(u, v) \approx (-\psi_y, \psi_x)$, the vorticity equation becomes

$$\nabla^2 \psi_t + \beta \psi_x = O(\delta)\,, \tag{4.13}$$

and we can ignore the right-hand side. Assuming the stream function has the wave-like structure of a spherical harmonic, we substitute the expression $\psi = \psi_0 Y_n^m(\lambda, \phi) \exp(-i\nu t)$ in the vorticity equation and immediately deduce an expression for the frequency:

$$\nu = \nu_R \equiv -\frac{2\Omega m}{n(n+1)}\,. \tag{4.14}$$

This is the celebrated dispersion relation for Rossby-Haurwitz waves (Haurwitz, 1940). If we ignore sphericity (the β-plane approximation) and assume harmonic dependence $\psi(x, y, t) = \psi_0 \exp[i(kx + \ell y - \nu t)]$, then (4.13) has the dispersion relation

$$c = \frac{\nu}{k} = -\frac{\beta}{k^2 + \ell^2}\,,$$

which is the expression for phase-speed found by Rossby (1939). The Rossby or Rossby-Haurwitz waves are, to the first approximation, non-divergent waves which travel westward, the phase speed being greatest for the waves of largest scale. They are of relatively low frequency — (4.14) implies that $|\nu| \leq \Omega$ — and the frequency decreases as the spatial scale decreases.

To the same degree of approximation, we may write the divergence equation (4.11) as

$$\nabla^2 P - f\zeta - \beta\psi_y = O(\delta)\,. \tag{4.15}$$

Ignoring the right-hand side, we get the *linear balance equation*

$$\nabla^2 P = \nabla \cdot f \nabla \psi\,, \tag{4.16}$$

a diagnostic relationship between the geopotential and the stream function. This also follows immediately from the assumption that the wind is both non-divergent ($\mathbf{V} = \mathbf{k} \times \nabla\psi$) and geostrophic ($f\mathbf{V} = \mathbf{k} \times \nabla P$). If variations of f are ignored, we can assume $P = f\psi$. The wind and pressure are in approximate geostrophic balance for Rossby-Haurwitz waves.

4.4. GRAVITY WAVE MODES

If we assume now that the solution is quasi-irrotational, *i.e.* that $|\zeta| \ll |\delta|$, then the wind is given approximately by $(u, v) \approx (\chi_x, \chi_y)$ and the divergence equation becomes

$$\nabla^2\chi_t + \beta\chi_x + \nabla^2 P = O(\zeta)$$

with the right-hand side negligible. Using the continuity equation to eliminate P, we get

$$\nabla^2\chi_{tt} + \beta\chi_{xt} - gh\nabla^4\chi = 0\,.$$

Seeking a solution $\chi = \chi_0 Y_n^m(\lambda, \phi)\exp(-i\nu t)$, we find that

$$\nu^2 + \left(-\frac{2\Omega m}{n(n+1)}\right)\nu - \frac{n(n+1)gh}{a^2} = 0\,. \tag{4.17}$$

The coefficient of the second term is just the Rossby-Haurwitz frequency ν_R found in (4.14) above, so that

$$\nu = \pm\sqrt{\nu_G^2 + (\frac{1}{2}\nu_R)^2} - \frac{1}{2}\nu_R\,, \qquad \text{where} \qquad \nu_G \equiv \sqrt{\frac{n(n+1)gh}{a^2}}\,,$$

Noting that $|\nu_G| \gg |\nu_R|$, it follows that

$$\nu_\pm \approx \pm\nu_G\,,$$

the frequency of pure gravity waves. There are then two solutions, representing waves travelling eastward and westward with equal speeds. The frequency increases approximately linearly with the total wavenumber n.

5. Normal Mode Initialization

The model equations can be written schematically in the form

$$\dot{\mathbf{X}} + i\mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = \mathbf{0} \tag{5.1}$$

with $\mathbf{X}$ the state vector, $\mathbf{L}$ a matrix and $\mathbf{N}$ a nonlinear vector function. If $\mathbf{L}$ is diagonalized, the system separates into two subsystems, for the low and high frequency components:

$$\dot{\mathbf{Y}} + i\Lambda_Y\mathbf{Y} + \mathbf{N}_Y(\mathbf{Y}, \mathbf{Z}) = \mathbf{0} \tag{5.2}$$

$$\dot{\mathbf{Z}} + i\Lambda_Z\mathbf{Z} + \mathbf{N}_Z(\mathbf{Y}, \mathbf{Z}) = \mathbf{0} \tag{5.3}$$

where $\mathbf{Y}$ and $\mathbf{Z}$ are the coefficients of the LF and HF components of the flow, referred to colloquially as the *slow* and *fast* components respectively, and Λ_Y and Λ_Z are diagonal matrices of eigenfrequencies for the two types of modes.

Let us suppose that the initial fields are separated into slow and fast parts, and that the latter are removed so as to leave only the Rossby waves. It might be hoped that this process of "linear normal mode initialization" (imposing the condition $\mathbf{Z} = \mathbf{0}$ at $t = 0$) would ensure a noise-free forecast. However, the results of the technique are disappointing: the noise is reduced initially, but soon reappears; the forecasting equations are nonlinear, and the slow components interact nonlinearly in such a way as to generate gravity waves. The problem of noise remains: the gravity waves are small to begin with, but they grow rapidly (see Daley, Ch. 9).

To control the growth of HF components, Machenhauer (1977) proposed setting their initial rate-of-change to zero, in the hope that they would remain small throughout the forecast. Baer (1977) proposed a somewhat more general method, using a two-timing perturbation technique. The forecast, starting from initial fields modified so that $\dot{\mathbf{Z}} = \mathbf{0}$ at $t = 0$ is very smooth and the spurious gravity wave oscillations are almost completely removed. The method takes account of the nonlinear nature of the equations, and is referred to as nonlinear normal mode initialization (NNMI). The method is comprehensively reviewed in Daley (1991). Rather than considering the full complexity of an atmospheric model, we will illustrate LNMI and NNMI by application to a simple mechanical system.

6. Initialization and the Swinging Spring

The procedure of linear and nonlinear normal mode initialization can be clearly illustrated by applying the method to the equations of the elastic pendulum or 'swinging spring'. This system comprises a heavy bob suspended by a light elastic spring. The bob is free to move in a vertical plane. The oscillations of this system are of two types, distinguished by their physical restoring mechanisms. For an appropriate choice of parameters, the elastic oscillations have much higher frequency than the rotation or libration of the bob. We consider the elastic oscillations to be analogues of the high frequency gravity waves in the atmosphere. Similarly, the low frequency rotational motions are considered to correspond to the rotational or Rossby-Haurwitz waves.

6.1. THE DYNAMICAL EQUATIONS

Let ℓ_0 be the unstretched length of the spring, k its elasticity or stiffness and m the mass of the bob. At equilibrium the elastic restoring force is balanced by the weight: $k(\ell - \ell_0) = mg$. Polar coordinates $q_r = r$ and $q_\theta = \theta$ are used, and the radial and angular momenta are $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$. The Hamiltonian is, in this case, the sum of kinetic, elastic potential and gravitational potential energy:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{1}{2}k(r-\ell_0)^2 - mgr\cos\theta\,.$$

The (canonical) dynamical equations may now be written explicitly

$$\begin{aligned}\dot\theta &= p_\theta/mr^2\\ \dot p_\theta &= -mgr\sin\theta\\ \dot r &= p_r/m\\ \dot p_r &= p_\theta^2/mr^3 - k(r-\ell_0) + mg\cos\theta\,.\end{aligned}$$

(If the Hamiltonian formalism is unfamiliar, the equations may be derived by considering the forces on the bob). These equations may also be written symbolically in vector form

$$\dot{\mathbf{X}} + \mathbf{L}\mathbf{X} + \mathbf{N}(\mathbf{X}) = 0$$

where $\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}}$, $\mathbf{L}$ is the matrix of coefficients of the linear terms and $\mathbf{N}$ is a nonlinear vector function.

Let us now suppose that the amplitude of the motion is small, so that $|r'| = |r-\ell| \ll \ell$ and $|\theta| \ll 1$. The state vector $\mathbf{X}$ comprises two sub-vectors:

$$\mathbf{X} = \begin{pmatrix}\mathbf{Y}\\ \mathbf{Z}\end{pmatrix}, \qquad \text{where} \quad \mathbf{Y} = \begin{pmatrix}\theta\\ p_\theta\end{pmatrix} \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix}r'\\ p_r\end{pmatrix},$$

and the linear dynamics of these components evolve independently. We call the motion described by $\mathbf{Y}$ the rotational component and that described by $\mathbf{Z}$ the elastic component. The rotational equations may be written

$$\ddot\theta + (g/\ell)\theta = 0$$

which is the equation for a simple pendulum having oscillatory solutions with frequency $\sqrt{g/\ell}$. The remaining two equations yield

$$\ddot r' + (k/m)r' = 0\,,$$

the equations for elastic oscillations with frequency $\sqrt{k/m}$. We define the rotational and elastic frequencies and their ratio by

$$\omega_{\mathrm{R}} = \sqrt{\frac{g}{\ell}}, \qquad \omega_{\mathrm{E}} = \sqrt{\frac{k}{m}}, \qquad \epsilon \equiv \left(\frac{\omega_{\mathrm{R}}}{\omega_{\mathrm{E}}}\right).$$

It is easily shown that $\epsilon < 1$, so the rotational frequency is always less than the elastic. We assume that the parameters are such that $\epsilon \ll 1$. In this case the linear normal modes are clearly distinct: the rotational mode has low frequency (LF) and the elastic mode has high frequency (HF).

6.2. LINEAR AND NONLINEAR INITIALIZATION

For small amplitude motions, for which the nonlinear terms are negligible, the LF and HF oscillations are completely independent of each other and evolve without interaction. We can suppress the HF component completely by setting its initial amplitude to zero:

$$\mathbf{Z} = (r', p_r)^{\mathrm{T}} = \mathbf{0} \quad \text{at} \quad t = 0\,.$$

This procedure is called linear initialization. When the amplitude is large, nonlinear terms are no longer negligible and the LF and HF motions interact. It is clear from the equations that linear initialization will not ensure permanent absence of HF motions: the nonlinear LF terms generate radial momentum. To achieve better results, we set the initial *tendency* of the HF components to zero:

$$\dot{\mathbf{Z}} = (\dot{r}, \dot{p}_r)^{\mathrm{T}} = \mathbf{0} \quad \text{at} \quad t = 0\,,$$

This procedure is called nonlinear initialization. For the spring, we can deduce explicit expressions for the initial conditions:

$$r(0) = r_{\mathrm{B}} \equiv \frac{\ell(1 - \epsilon^2(1 - \cos\theta))}{1 - (\dot{\theta}/\omega_{\mathrm{E}})^2}\,, \qquad p_r(0) = 0\,.$$

Thus, given arbitrary initial conditions $\mathbf{X} = (\theta, p_\theta, r, p_r)^{\mathrm{T}}$, we replace $\mathbf{Z} = (r, p_r)^{\mathrm{T}}$ by $\mathbf{Z}_{\mathrm{B}} = (r_{\mathrm{B}}, 0)^{\mathrm{T}}$. The rotational component $\mathbf{Y} = (\theta, p_\theta)^{\mathrm{T}}$ remains unchanged.

6.3. A NUMERICAL EXAMPLE

In Figure 1 we show the results of two integrations of the spring equations. The upper panels show the evolution and spectrum of the slow variable θ; the lower panels are for the fast variable r. Dotted curves are for linear initialization and solid curves for nonlinear initialization. The parameter values are $m = 1$, $g = \pi^2$, $k = 100\pi^2$ and $\ell = 1$ (all SI units), so that $\epsilon = 0.1$ and the periods of the swinging and springing motions are respectively $\tau_{\mathrm{R}} =$ 2s and $\tau_{\mathrm{E}} = 0.2$s. The initial conditions are vanishing velocity ($\dot{r} = \dot{\theta} = 0$), with $\theta(0) = 1$ and $r(0) \in \{1, 0.99540\}$. The equations are integrated over a period of 6 seconds For the slow variable, the curves are indistinguishable. The spectrum has a clear peak at a frequency of 0.5 cycles per second (Hz). For the fast variable, the linearly initialized evolution has high frequency noise (dotted curve, lower left panel). This is confirmed in the spectrum: there is a sharp peak at 5 Hz. When nonlinearly initialized, this peak is removed: only the peak at 1 Hz remains. This is the 'balanced fast motion'. It can be understood physically: the centrifugal effect stretches the spring twice for each pendular swing: the result is a component of r with a period of one second. The radial variation does not disappear for balanced motion, but it is of low frequency. It is said to be 'slaved' (or, better, enslaved) to the slow motion.

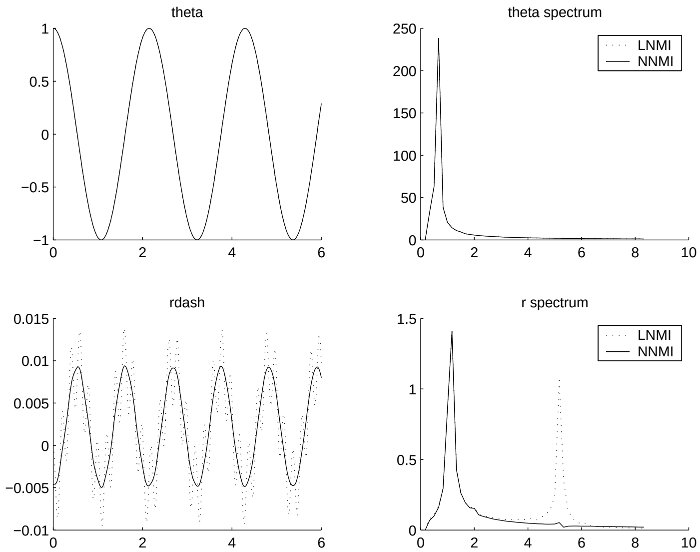

Figure 1. Solution of swinging spring equations for linear (LNMI) and nonlinear (NNMI) initialization. See text for details.

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