Isogenies and the Discrete Logarithm Problem in Genus Three

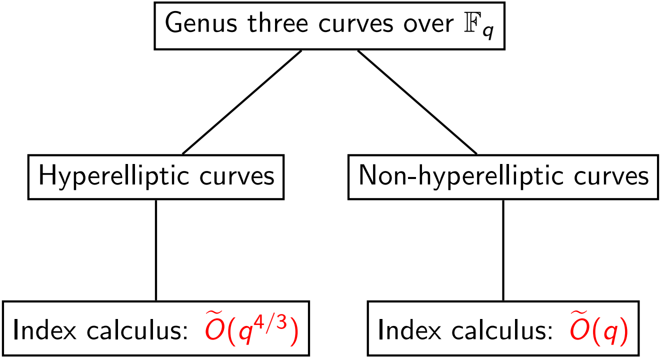
Benjamin Smith

Royal Holloway, University of London

September 6, 2007

q is odd!

Curves of genus three



Index calculus on hyperelliptic curves: Gaudry–Thomé–Theriault–Diem
Index calculus on non-hyperelliptic curves: Diem

Hyperelliptic and non-hyperelliptic curves of genus three

Hyperelliptic curves $H/\mathbb{F}_q$:
Defining equation:

$$H : y^2 = F(x, z),$$

where F is a squarefree homogeneous polynomial of degree 8
($\longrightarrow$ projective model in $\mathbb{P}(1, 4, 1)$).
Canonical map: $\pi : H \longrightarrow \mathbb{P}^1$, $(x : y : z) \longmapsto (x : z)$.
Involution: $\iota : (x : y : z) \mapsto (x : -y : z)$.

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Non-hyperelliptic curves $C/\mathbb{F}_q$:
Defining equation:

$$C : F(x, y, z) = 0,$$

where F is a homogeneous polynomial of degree 4
(Plane Quartic Model in $\mathbb{P}^2$).
Canonical map: embedding $C \hookrightarrow \mathbb{P}^2$.

More on genus three curves

Throughout, we adopt these conventions:

- X always denotes a curve of genus three
- H always denotes a hyperelliptic curve of genus three
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More on genus three curves

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The **Jacobian** J_X of X is a three-dimensional PPAV.

Points of J_X correspond to divisor classes on X (elements of $\mathrm{Pic}^0(X)$); that is, equivalence classes of formal sums $\sum_i P_i$ of points on X.

Nonsingular projective embeddings of J_X are too hard to work with, so we always work with $\mathrm{Pic}^0(X)$ and X instead.

Homomorphisms and the DLP

Hyperelliptic and non-hyperelliptic curves
have completely different geometries.

H **cannot** be isomorphic to C

$\implies$ J_H **cannot** be isomorphic to J_C (as PPAVs)

...so we can't translate Index Calculus algorithms between J_C and J_H.

But we **can** have homomorphisms $J_H \to J_C$
— so we should be able to translate DLPs from J_H to J_C:

$$Q = [m]P \implies \phi(Q) = [m]\phi(P).$$

A surjective homomorphism with finite kernel is called an **isogeny**.

Our aim

Aim: explicit isogenies from hyperelliptic to non-hyperelliptic Jacobians.

Oort and Ueno:

every 3-dimensional PPAV is isomorphic (over $\mathbb{F}_{q^2}$) to a Jacobian.

$\Longrightarrow$ quotients of J_H by small subgroups give isogenies to other Jacobians.

Naïve picture of moduli spaces:

(It's on the board!)

If we start from J_H and take an arbitrary isogeny $J_H \to J_X$,
then with overwhelming probability we will have an isomorphism $X \cong C$,
and hence an isogeny $J_H \to J_C$.

Computing explicit isogenies

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The **Weierstrass points** of $H : y^2 = F(x,z)$
are the eight points $W_1, \ldots, W_8$ of $H(\overline{\mathbb{F}_q})$ where $y(W_i) = 0$.

The divisor classes $[W_1 - W_2]$, $[W_3 - W_4]$, $[W_5 - W_6]$, and $[W_7 - W_8]$
generate a subgroup $S \cong (\mathbb{Z}/2\mathbb{Z})^3$ of J_H.
We call such subgroups **tractable subgroups**.

We have derived algorithms to compute isogenies with tractable kernels.

Geometric methods

Suppose we are given H and $S = \langle [W_i - W_{i+1}] : i \in \{1, 3, 5, 7\} \rangle$.

Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be a 3-to-1 (**trigonal**) map such that

$$g(W_i) = g(W_{i+1}) \text{ for each } [W_i - W_{i+1}] \in S.$$

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Recillas' **trigonal construction**, applied to $\pi : H \to \mathbb{P}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$, yields a curve X of genus three and a 4-to-1 map $f : X \to \mathbb{P}^1$.
Donagi and Livné: there is an isogeny $\phi : J_H \to J_X$ with kernel S.

If Q is a point on $\mathbb{P}^1$, then

$$(g \circ \pi)^{-1}(Q) = \{P_1, P_2, P_3, \iota(P_1), \iota(P_2), \iota(P_3)\}$$

$$f^{-1}(Q) = \left\{ \begin{array}{c} Q_1 \leftrightarrow \{P_1, P_2, P_3 | \iota(P_1), \iota(P_2), \iota(P_3)\}, \\ Q_2 \leftrightarrow \{P_1, \iota(P_2), \iota(P_3) | \iota(P_1), P_2, P_3\}, \\ Q_3 \leftrightarrow \{\iota(P_1), P_2, \iota(P_3) | P_1, \iota(P_2), P_3\}, \\ Q_4 \leftrightarrow \{\iota(P_1), \iota(P_2), P_3 | P_1, P_2, \iota(P_3)\} \end{array} \right\}$$

Everybody loves commutative diagrams...

$$
\begin{tikzcd}
 & R \arrow[dl, "\pi_H"', "2"] \arrow[dr, "\pi_X", "3"'] & & \\
H \arrow[d, "\pi"', "2"] & & X \arrow[r, "\cong", "??"'] \arrow[ddl, "4"', "f"'] & C \\
\mathbb{P}^1 \arrow[dr, "3", "g"'] & & & \\
 & \mathbb{P}^1 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & J_R \arrow[dr, "(\pi_X)_*"] & & \\
J_H \arrow[ur, "(\pi_H)^*"] \arrow[rrr, bend right, "\phi"'] & & J_X \arrow[r, "\cong"] & J_C
\end{tikzcd}
$$

Explicit trigonal constructions

Given S (over $\mathbb{F}_q$), we can compute g using basic linear algebra.
this requires solving a quadratic equation over $\mathbb{F}_q$.

Given g and H, we can compute a model of X in $\mathbb{A}^1 \times \mathbb{A}^3$ using linear algebra and modular polynomial arithmetic. (The computation is involved, but essentially easy.)
Again, we need to solve a quadratic equation over $\mathbb{F}_q$.

The map $f : X \to \mathbb{A}^1$ is projection onto the first factor.

Having computed g, f, and X, we get R, π_H and π_X "for free".

Finally, the canonical map of X (for the isomorphism to C) can be computed quickly using standard algorithms.

Rationality

It is important that our isogenies be $\mathbb{F}_q$-rational
— otherwise they map $J_H(\mathbb{F}_q)$ into $J_C(\mathbb{F}_{q^d})$;
Index Calculus in $J_C(\mathbb{F}_{q^d})$ requires $\widetilde{O}(q^d)$ time, so we gain nothing!

We therefore need

1. A rational kernel subgroup S
2. A rational trigonal map g
   $\longrightarrow 1/2$ probability for a given rational S
3. A rational model for X
   $\longrightarrow 1/2$ probability for a given rational S and g

We should be able to use descent to deal with irrational trigonal maps g.

How many kernel subgroups are there?

$H : y^2 = F(x, z)$, F homogeneous, squarefree, $\deg F = 8$.
$\mathcal{S}(H) :=$ set of $\mathbb{F}_q$-rational tractable subgroups of J_H.

Degrees of k-irreducible factors of F	$\#\mathcal{S}(H)$
$(8), (6, 2), (6, 1, 1), (4, 2, 1, 1)$	1
$(4, 4)$	5
$(4, 2, 2), (4, 1, 1, 1, 1), (3, 3, 2), (3, 3, 1, 1)$	3
$(2, 2, 2, 1, 1)$	7
$(2, 2, 1, 1, 1, 1)$	9
$(2, 1, 1, 1, 1, 1, 1)$	15
$(2, 2, 2, 2)$	25
$(1, 1, 1, 1, 1, 1, 1, 1)$	105
Other	0

How often do we have a rational isogeny?

Summing over probabilities of the different factorization types, we find that for a randomly chosen $H : y^2 = F(x, z)$, there is an expectation of

$$\sim 18.57\%$$

that our methods will produce a rational isogeny from $J_H \to J_C$.

If we can use descent to account for the square root in computing g, we obtain an even better expectation:

$$\sim 31.13\%$$

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5. ...and probably will not work in higher genus either.
6. As things stand, "security" of genus three hyperelliptic Jacobians depends on the factorization of the hyperelliptic polynomial.

Thanks

Thanks: to Roger Oyono and Christophe Ritzenthaler